2.2 Propagator as Green function.
Show that the Green function or propagator $K(x',t';x,t) = \langle x'|\exp[-\mathrm{i}H(t'-t)]|x\rangle$ of the Schrödinger equation is the inverse of the differential operator $(\mathrm{i}\partial_t - H)$. Hint: It is simpler to consider the advanced or retarded Green functions,

$$G^{\pm}(x',t';x,t) = G(x',t';x,t)\vartheta(\pm\tau) \tag{24}$$

where $\tau = t' - t$ and ϑ is the step function.

The interpretation of the propagator as the Green's function G of the differential equation $(i\partial_t - H)$ is derived simplest after introducing the advanced and retarded Green's functions,

$$G^{\pm}(x',t';x,t) = G(x',t';x,t)\vartheta(\pm\tau) \tag{25}$$

where $\tau = t' - t$. [1] Considering e.g. the retarded case,

$$\vartheta(t'-t)\psi(x',t') = \mathrm{i}\int d^3x G^+(x',t';x,t)\psi(x,t) \tag{26}$$

applying the Schrödinger operator $[\mathrm{i}\partial_{t'} - H(x')]$ on it, gives

$$\left[\mathrm{i}\partial_{t'} - H(x')\right]\vartheta(t'-t)\psi(x',t') = \mathrm{i}\int d^3x[\mathrm{i}\partial_{t'} - H(x')]G^+(x',t';x,t)\psi(x,t)\,. \tag{27}$$

The LHS is

$$[\mathrm{i}\partial_{t'}\vartheta(t'-t)]\psi(x',t') + \vartheta(t'-t)\underbrace{[\mathrm{i}\partial_{t'} - H(x')]\psi(x',t')}_{=0} = \mathrm{i}\delta(t'-t)\psi(x',t) \tag{28}$$

where $\vartheta' = \delta$ and the Schrödinger equation were used.
Thus

$$\mathrm{i}\delta(t'-t)\psi(x',t) = \mathrm{i}\int d^3x[\mathrm{i}\partial_{t'} - H(x')]G^+(x',t';x,t)\psi(x,t) \tag{29}$$

can be valid for an arbitrary solution ψ only if

$$[\mathrm{i}\partial_{t'} - H(x')]G^+(x',t';x,t) = \delta^{(4)}(x-x')\,. \tag{30}$$

Thus the retarded Green's functions G^+ is the inverse of the differential operator $(i\partial_t - H)$. The same reasoning holds for the advanced Green function G^-, and thus also for $G = G^+ + G^-$.

[1] The step function $\vartheta(x) = 0$ for $x < 0$ and $\vartheta(x) = 1$ for $x > 0$. From its integral representation $\vartheta(\tau) = -\frac{1}{2\pi\mathrm{i}}\int d\omega \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega + \mathrm{i}\varepsilon}$ one derives $\vartheta' = \delta$.