## 2.9 Propagator at large $|\boldsymbol{x}|$.

Show that the propagator $K(\boldsymbol{x},0;0,0) = \mathrm{i}\Delta_F(0,r$ decays exponentially outside the light-cone.

We start from

$$\mathrm{i}\Delta_F(0,r) = \int \frac{\mathrm{d}^3k}{2\omega_k(2\pi)^3}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{kx}} = \frac{1}{2(2\pi)^2}\int_0^\infty \frac{\mathrm{d}k\,k^2}{\sqrt{k^2+m^2}}\int_{-1}^1 \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}krz} \tag{35}$$

where $z = \cos\vartheta$ is the angle between $\boldsymbol{k}$ and $\boldsymbol{x}$. The integrand in

$$\Delta_F(0,r) = -\frac{1}{2(2\pi)^2 r}\int_0^\infty \frac{\mathrm{d}k\,k}{\sqrt{k^2+m^2}}\left(\mathrm{e}^{\mathrm{i}kr} - \mathrm{e}^{-\mathrm{i}kr}\right) \tag{36}$$

is $\propto k\sin(kr)$, i.e. even. Thus we can first extend it to $-\infty$, then add the odd function $\propto k\cos(kr)$ and get

$$\Delta_F(0,r) = -\frac{1}{8\pi^2 r}\int_\infty^\infty \frac{\mathrm{d}k\,k}{\sqrt{k^2+m^2}}\,\mathrm{e}^{\mathrm{i}kr} = \frac{\mathrm{i}}{8\pi^2 r}\frac{\partial}{\partial r}\int_{-\infty}^\infty \frac{\mathrm{d}k}{\sqrt{k^2+m^2}}\,\mathrm{e}^{\mathrm{i}kr}\,. \tag{37}$$

We work now on the last integral. Its has a cut from $\mathrm{i}k = m$ to $\mathrm{i}\infty$; We change $k = \mathrm{i}(m+y)$ and integrate on both sides of the cut,

$$\int_{-\infty}^\infty \frac{\mathrm{d}k\,k}{\sqrt{k^2+m^2}}\,\mathrm{e}^{\mathrm{i}kr} = 2\int_0^\infty \mathrm{d}y\frac{\mathrm{e}^{-m(m+y)r}}{\sqrt{(m^2+y^2)-y^2}} = 2\int_0^\infty \mathrm{d}u\frac{\mathrm{e}^{-mur}}{\sqrt{u^2-1}} = \tag{38}$$

$$= 2\int_0^\infty \mathrm{d}t\mathrm{e}^{-mr\cosh t} = 2K_0(mr)\,. \tag{39}$$

In the last step, we used e.g. 3.547(4) from Abramowitz& Stegun. Next we have to perform the derivative, using $K_0'(mr) = -mK_1(mr)$. Hence the final result is

$$\mathrm{i}\Delta_F(0,r) = \frac{m}{4\pi^2 r}K_1(mr)\,,$$

i.e. the propagator decays exponentially for space-like separations. The result for time-like separations follows by analytic continuation.