24.14 Jeans length and the sound speed.

We start with the Euler equations

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\boldsymbol{u} \quad \rightarrow \quad \frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot(\nabla\rho) + \rho(\nabla\cdot\boldsymbol{u}) = 0 \tag{322}$$

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla\left(\frac{P}{\rho}\right) + \boldsymbol{g} \quad \rightarrow \quad \frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot(\nabla\boldsymbol{u}) + \frac{\nabla P}{\rho} = \boldsymbol{g}. \tag{323}$$

With the perturbation ansatz (small perturbations in a fluid at rest)

$$\rho = \rho_0 + \varepsilon\rho_1(x,t) \tag{324}$$

$$P = P_0 + \varepsilon P_1(x,t) \tag{325}$$

$$\boldsymbol{u} = \varepsilon\boldsymbol{u}_1(x,t) \tag{326}$$

and the Poisson equation

$$\triangle\phi = 4\pi G\rho \quad \rightarrow \quad \nabla\cdot\boldsymbol{g}_1 = -4\pi G\rho_1 \tag{327}$$

we obtain (with the EoS $P = w\rho$) in order ε

$$\frac{\partial\rho_1}{\partial t} + \rho_0(\nabla\cdot\boldsymbol{u}_1) = 0 \tag{328}$$

$$\frac{\partial\boldsymbol{u}_1}{\partial t} + \underbrace{\frac{1}{\rho_0}\nabla P_1}_{=\frac{w}{\rho_0}\nabla\rho_1} = \boldsymbol{g}_1. \tag{329}$$

Differentiating both (with respect to space and time) we obtain a wave equation

$$\frac{\partial^2\rho_1}{\partial t^2} - w\triangle\rho_1 = 4\pi G\rho_0\rho_1 \tag{330}$$

with the speed of sound $c_s^2 = w$. Inserting the wave ansatz $\rho_1 \sim \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$ yields the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0. \tag{331}$$

For wave numbers $k_J < \sqrt{4\pi G/c_s^2}$ the ω becomes complex which gives rise to exponentially growing modes. Therefore the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J} = c_s\sqrt{\frac{\pi}{G\rho_0}} = \sqrt{\frac{\pi w}{G\rho_0}}. \tag{332}$$