25.2 Orbits of massive particles.

a. Spherical symmetry allows us to choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. Then we replace in the normalization condition $\boldsymbol{u}\cdot\boldsymbol{u} = 1$ written out for the Schwarzschild metric,

$$1 = \left(1-\frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2$$

the velocities u_t and u_r by the conserved quantities

$$\begin{aligned} e &\equiv \boldsymbol{\xi}\cdot\mathbf{u} = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ l &\equiv \boldsymbol{\eta}\cdot\mathbf{u} = r^2\sin\vartheta^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = r^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,. \end{aligned}$$

Inserting e and l, then reordering gives

$$\mathcal{E} \equiv \frac{e^2-1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r) \tag{310}$$

with

$$V_{\text{eff}}(r) = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}\,.$$

Eq. (310) has the same form as in Newtonian gravity, with the only change that the effective potential contains the general relativistic correction term $-Ml^2/r^3$ (suppressed by $1/c^2$) that becomes important at small r. Thus the analysis of possible orbits proceeds as in Newtonian gravity. Note however the different meaning of the variable r in the two cases: In the Newtonian case, the coordinate r equals the physical distance to the star, in the general relativistic case the coordinate r and the distance differ.

b.) The potential is shown in Fig. 1. The asymptotic behavior of V_{eff} for $r\to 0$ and $r\to\infty$ is

$$V_{\text{eff}}(r\to\infty) \to -\frac{M}{r} \qquad \text{and} \quad V_{\text{eff}}(r\to 0) \to -\frac{Ml^2}{r^3}\,, \tag{311}$$

while the potential at the Schwarschild radius, $V(2M) = 1/2$, is independent of M.
We determine the extrema of V_{eff} by solving $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ and find

$$r_{1,2} = \frac{l^2}{2M}\left[1 \pm \sqrt{1-12M^2/l^2}\right] \tag{312}$$

Hence the potential has no extrema for $l/M > \sqrt{12}$ and is always negative: A particle can reach $r = 0$ for small enough but finite angular momentum, in contrast to the Newtonian case. By the same argument, there exists a last stable orbit at $r = 6M$, when the two extrema r_1 and r_2 coincide for $l/M = \sqrt{12}$. This so-called "innermost stable circular orbit" at $r_{\text{ISCO}} = 6M$ for a Schwarzschild black hole is e.g. relevant for the study of accretion disks around black holes.
The orbits can be classified according the relative size of $\mathcal{E}$ and V_{eff} for a given l:

- Bound orbit exists for $\mathcal{E} < 0$. Two circular orbits, one stable at the minimum of V_{eff} and an unstable one at the maximum of V_{eff}; non-circular orbits oscillate between two turning points.
- Scattering orbit exists for $\mathcal{E} > 0$: If $\mathcal{E} > \max\{V_{\text{eff}}\}$, the particle hits after a finite time the singularity $r = 0$. For $0 < \mathcal{E} < \max\{V_{\text{eff}}\}$, the particle turns at $\mathcal{E} = \max\{V_{\text{eff}}\}$ and escapes to $r\to\infty$.

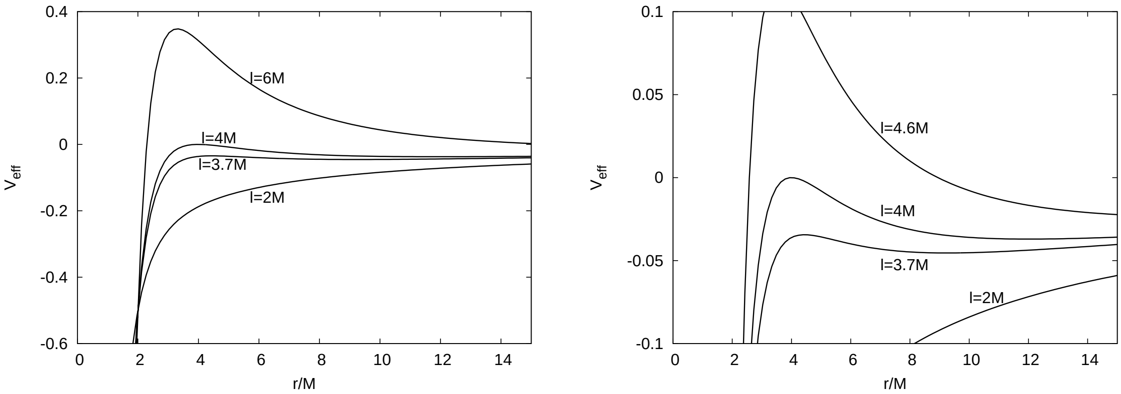

Figure 1: The effective potential V_{eff} for various values of l/M as function of distance r/M, for two different scales.