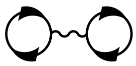
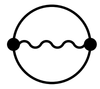
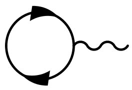
26.2 Vacuum energy in QED.
Consider the vacuum energy in QED at two loop. Argue that one of the two diagrams vanishes for a charge-neutral vacuum. Connect the other one to the one-loop photon polarisation $\Pi_{\mu\nu}$ and use our old results to calculate it.

The vacuum diagrams in QED at two-loop order are given by

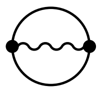
We can view each half of the left diagram as the vev of the current operator j^μ, with $\langle j^0 \rangle$ as the charge of the vacuum. Since the vacuum should be neutral, we expect that $\langle j^0 \rangle$ and thus $\langle j^\mu \rangle$ vanishes. This is guranteed by the Furry theorem which informs us that indeed the left diagram vanishes. Alternatively, we see by direct calculation that



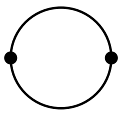
$$= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\operatorname{tr}[\gamma^\mu(\not{p}+m)]}{p^2-m^2+\mathrm{i}\varepsilon} = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{4p^\mu}{p^2-m^2+\mathrm{i}\varepsilon} = 0\,, \tag{340}$$

because the integrand is antisymmetric.

Next we note that we can write the right diagram as



$$= \frac{1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{-\mathrm{i}(\eta^{\mu\nu}-(1-\xi)q^\mu q^\nu/q^2)}{q^2+\mathrm{i}\varepsilon} \times \tag{341}$$



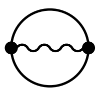
with $S=1/2$ as the relative symmetry factor of the two vacuum diagram. Then we express the vacuum polarisation as

$$\Pi_{\mu\nu}(q^2) = (q^2\eta_{\mu\nu} - q_\mu q_\nu)\Pi(q^2)\,,$$

use $\eta^{\mu\nu}\eta_{\mu\nu} = d$ and obtain

$$-\mathrm{i}(\eta^{\mu\nu}-(1-\xi)q^\mu q^\nu/q^2)(q^2\eta_{\mu\nu}-q_\mu q_\nu)\mathrm{i}\Pi(q^2) = (d-1)q^2\Pi(q^2)\,. \tag{342}$$

We found thus



$$= \frac{d-1}{2}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\,\Pi(q^2) \equiv -\mathrm{i}\rho^{(2)}_{\rm QED}\,. \tag{343}$$

Finally we insert

$$\Pi(q^2) = A\Gamma(2-d/2)\int_0^1 \mathrm{d}x\, x(1-x)\,[m^2+x(1-x)q^2]^{-2+d/2} \tag{344}$$

substitute $t^2 = x(1-x)q^2$ and evaluate the integrals,

$$\rho^{(2)}_{\mathrm{QED}} = \frac{12e^2}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 \mathrm{d}x\,x(1-x)\int\frac{\mathrm{d}^d q}{(2\pi)^d}\,[m^2+x(1-x)q^2]^{-2+d/2} \tag{345}$$

$$= \frac{12e^2}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 \mathrm{d}x\,[x(1-x)]^{1-d/2}\int\frac{\mathrm{d}^d t}{(2\pi)^d}\,[m^2+t^2]^{-2+d/2} \tag{346}$$

$$= \frac{12e^2}{(4\pi)^{d}}\,\Gamma(2-d)\int_0^1 \mathrm{d}x\,[x(1-x)]^{1-d/2}\,m^{2d-4} \tag{347}$$

$$= \frac{12e^2}{(4\pi)^{d}}\,\frac{\Gamma(2-d/2)^2}{(3-d)(2-d)}\,m^{2d-4}\,. \tag{348}$$

Thus the 2-loop vacuum energy in QED is proportional to $e^2\sum_f q_f^2 m_f^4$.