26.3 Fixed points of a $\Omega_m + \Omega_\Lambda = 1$ universe.

a.) Multiply the acceleration equation by $a/\dot{a}^2$ and introduce w_i,

$$\frac{a\ddot{a}}{\dot{a}^2} = -\frac{4\pi G}{3} H^2 \sum_i \rho_i (1+w_i).$$

The LHS equals minus $q(t)$. Using the definition of $\rho_{\rm cr}$ and inserting w_i, it follows

$$q = \frac{1}{2}(\Omega_m + 2\Omega_r - 2\Omega_\Lambda). \tag{344}$$

b.) We differentiate Ω_i,

$$\dot{\Omega}_i = \frac{8\pi G}{3H^2}\left(\dot{\rho}_i - 2\frac{\dot{H}}{H}\rho_i\right).$$

Combined with $\dot{\rho}_i = -3(1+w_i)H\rho_i$, we obtain

$$\dot{\Omega}_i = -\Omega_i\left(3(1+w_i) + 2\frac{\dot{H}}{H^2}\right)H.$$

Next we differentiate $H = \dot{a}/a$, obtaining $\dot{H} = \ddot{a}/a - H^2$ or

$$\frac{\dot{H}}{H^2} = \frac{a\ddot{a}}{\dot{a}^2} - 1 = -(1+q).$$

Replacing q by (344), it follows

$$\dot{\Omega}_i = \Omega_i H(\Omega_m + 2\Omega_r - 2\Omega_\Lambda - 1 - 3w_i)\,.$$

c.) Setting $\Omega_r = 0$ and inserting $w_m = 0$ and $w_\Lambda = -1$ gives

$$\dot{\Omega}_m = \Omega_m H[(\Omega_m - 1) - 2\Omega_\Lambda] \quad \text{and} \quad \dot{\Omega}_\Lambda = \Omega_\Lambda H[\Omega_m - 2(\Omega_\Lambda - 2)].$$

We elimininate H and t dividing the two equations, obtaining a first-order differntial equation,

$$\frac{\mathrm{d}\Omega_\Lambda}{\mathrm{d}\Omega_m} = \frac{\Omega_\Lambda[\Omega_m - 2(\Omega_\Lambda - 2)]}{\Omega_m[(\Omega_m - 1) - 2\Omega_\Lambda]}$$

Trajectories in the $(\Omega_\Lambda, \Omega_m)$ plane describe the evolution of a specific universe (which could be specified e.g. by the set of present day values $(\Omega_{\Lambda,0}, \Omega_{m,0})$). The condition $\frac{\mathrm{d}\Omega_\Lambda}{\mathrm{d}\Omega_m} = 0$ specifies fixed points: The point (0,1) is an unstable fixed point (starting point of all trajectories), while (1,0) is a stable fixed point (where all trajectoires end).