26.4 Fixed points of (26.18).

We start with

$$\text{(F1)} \qquad H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V + \rho\right) \tag{373}$$

$$\text{(F2)} \qquad \dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{374}$$

$$\text{(KG)} \qquad \ddot{\phi} = -3H\dot{\phi} - V_{,\phi}. \tag{375}$$

Using $H = \dot{a}/a$, $N = \ln(a)$ and $\lambda = -V_{,\phi}/(\sqrt{8\pi G}V)$ we obtain for the time derivatives of x and y

$$\dot{V} = \frac{dV}{d\phi}\frac{d\phi}{dt} = V_{,\phi}\dot{\phi} \tag{376}$$

$$x = \sqrt{\frac{4}{3}\pi G}\frac{\dot{\phi}}{H} \quad \to \quad \frac{dx}{dt} = \frac{dx}{dN}\frac{d\ln(a)}{dt} = \frac{dx}{dN}H = \sqrt{\frac{4}{3}\pi G}\frac{\ddot{\phi}H - \dot{\phi}\dot{H}}{H^2} \tag{377}$$

$$y = \sqrt{\frac{8}{3}\pi G}\frac{\sqrt{V}}{H} \quad \to \quad \frac{dy}{dt} = \frac{dy}{dN}\frac{d\ln(a)}{dt} = \frac{dy}{dN}H = \sqrt{\frac{8}{3}\pi G}\frac{\frac{V_{,\phi}\dot{\phi}}{2\sqrt{V}} - \sqrt{V}\dot{H}}{H^2}. \tag{378}$$

With the substitutions

$$\dot{H} = -4\pi G\left[\dot{\phi}^2 + (1+w_m)\rho\right] \tag{379}$$

$$\ddot{\phi} = -3H\dot{\phi} - V_{,\phi} \tag{380}$$

$$V_{,\phi} = -\sqrt{8\pi G}\lambda V \tag{381}$$

$$\rho = \frac{3H^2}{8\pi G} - \frac{1}{2}\dot{\phi}^2 - V \tag{382}$$

$$\dot{\phi} = xH/\sqrt{\frac{4}{3}\pi G} \tag{383}$$

$$\sqrt{V} = yH/\sqrt{\frac{8}{3}\pi G} \tag{384}$$

we obtain

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] \tag{385}$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)]. \tag{386}$$

To find the fix points of (26.17) we need to solve

$$-3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0 \tag{387}$$

$$-\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y[(1-w_m)x^2 + (1+w_m)(1-y^2)] = 0. \tag{388}$$

- An obvious solution is

$$x_0 = 0, y_0 = 0. \tag{389}$$

- Two semi-obvious solutions can be found for $y = 0$ which solves the second equation and transforms the first to the quadratic equation $x^2 - 1 = 0$ which gives

$$x_1 = +1, y_1 = 0 \tag{390}$$
$$x_2 = -1, y_2 = 0. \tag{391}$$

- Substituting the square bracket of the second equation into the first and simplifying the second gives

$$-3x + \frac{\sqrt{6}}{2}\lambda(x^2 + y^2) = 0 \tag{392}$$
$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}[1 + 2x^2 - (x^2 + y^2) - w_m((x^2 + x^2) - 1)] = 0. \tag{393}$$

Now we can eliminate $x^2 + y^2$ and obtain a single quadratic equation in x

$$-\frac{\sqrt{6}}{2}\lambda x + \frac{3}{2}\left[1 + 2x^2 - \frac{\sqrt{6}}{\lambda}x - w_m\left(\frac{\sqrt{6}}{\lambda}x - 1\right)\right] = 0 \tag{394}$$

which can be simplified to

$$x^2 - \frac{3(1 + w_m) + \lambda^2}{\sqrt{6}\lambda}x + \frac{1 + w_m}{2} = 0. \tag{395}$$

This gives us two more solutions

$$x_3 = \frac{\lambda}{\sqrt{6}}, y_3 = \sqrt{1 - \frac{\lambda^2}{6}} \qquad (\lambda^2 < 6) \tag{396}$$
$$x_4 = \sqrt{\frac{3}{2}}\frac{1 + w_m}{\lambda}, y_4 = \sqrt{\frac{3}{2}}\frac{\sqrt{1 - w_m^2}}{\lambda} \qquad (w_m^2 < 1). \tag{397}$$

- Let's quickly check the stability of the fix points. The characteristic equation for the fix points of a 2d system is given by

$$\alpha^2 + a_1(x_i, y_i)\alpha + a_2(x_i, y_i) = 0 \tag{398}$$
$$a_1(x_i, y_i) = -\left(\frac{df_x}{dx} + \frac{df_y}{dy}\right)_{x=x_i, y=y_i} \tag{399}$$
$$a_2(x_i, y_i) = \left.\frac{df_x}{dx}\frac{df_y}{dy} - \frac{df_x}{dy}\frac{df_y}{dx}\right|_{x=x_i, y=y_i} \tag{400}$$

with the stability classification (assuming for EoS parameter $w_m^2 < 1$)

type	condition	fix point 0	fix point 1	fix point 2
saddle node	$a_2 < 0$	$-1 < w_m < 1$	$\lambda > \sqrt{6}$	$\lambda < -\sqrt{6}$
unstable node	$0 < a_2 < a_1^2/4$	-	$\lambda < \sqrt{6}$	$\lambda > -\sqrt{6}$
unstable spiral	$a_1^2/4 < a_2, a_1 < 0$	-	-	-
center	$0 < a_2, a_1 = 0$	-	-	-
stable spiral	$a_1^2/4 < a_2, a_1 > 0$	-	-	-
stable node	$0 < a_2 < a_1^2/4$	-	-	-

type	fix point 3	fix point 4
saddle node	$3(1 + w_m) < \lambda^2 < 6$	-
unstable node	-	-
unstable spiral	-	-
center	-	-
stable spiral	-	$\lambda^2 > \frac{24(1+w_m)^2}{7+9w_m}$
stable node	$\lambda^2 < 3(1 + w_m)$	$\lambda^2 < \frac{24(1+w_m)^2}{7+9w_m}$