3.2 Maxwell Lagrangian.
a.) Derive the Lagrangian for the photon field A_μ from the source-free Maxwell equation $\partial_\mu F^{\mu\nu} = 0$. b.) What is the meaning of the unused set of Maxwell equations?

a.) We multiply the free field equation $\partial_\mu F^{\mu\nu} = 0$ by a variation δA that vanishes on the boundary $\partial\Omega$ of $\Omega = V \times [t_a : t_b]$. Then we integrate over Ω, and perform a partial integration,

$$\int_\Omega \mathrm{d}^4x\, \partial_\mu F^{\mu\nu}\ \delta A_\nu = -\int_\Omega \mathrm{d}^4x\, F^{\mu\nu}\ \delta(\partial_\mu A_\nu) = 0\,. \tag{41}$$

Next we note that

$$(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = 2(\partial_\alpha A_\beta - \partial_\beta A_\alpha)\partial^\alpha A^\beta \tag{42}$$

and thus

$$F^{\mu\nu}\ \delta(\partial_\mu A_\nu) = \frac{1}{2} F^{\mu\nu}\ \delta F_{\mu\nu}\,. \tag{43}$$

Applying the product rule, we obtain as final result

$$-\frac{1}{4}\,\delta \int_\Omega \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu} = \delta S[A_\mu] = 0 \tag{44}$$

and thus

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,. \tag{45}$$

Note that we expressed $\mathscr{L}$ trough F, but $\mathscr{L}$ should be viewed nevertheless as function of A. In order to convince us that the sign of the Lagrangian is correct, we would have to calculate the Hamitonian density $\mathscr{H}$ and to show that it is bounded from below.

b.) The other (homogeneous) set of Maxwell equations corresponds to

$$\partial_\alpha F^{\beta\gamma} + \partial_\beta F^{\gamma\alpha} + \partial_\gamma F^{\alpha\beta} = 0\,. \tag{46}$$

Introducing the dual field-strength tensor $\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ with $\varepsilon^{\alpha\beta\gamma\delta}$ as the completely anti-symmetric tensor, we can rewrite this (noting that all three terms are even permutations of the indices) as

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0\,. \tag{47}$$

We can view Eqs. (46-47) as the condition which allows us the express the field-strength tensor by the potential, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Having done this, any A_μ satisfies Eq. (46),

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma A_\delta = 0\,, \tag{48}$$

since we contract a symmetric tensor ($\partial_\alpha\partial_\gamma$) with an anti-symmetric one ($\varepsilon^{\alpha\beta\gamma\delta}$).
This is analogue to the familiar case of a "conservative" vector field V_μ: If V_μ is roatation-free, we can express it as a gradient of a scalar ϕ . On the other hand, any vector field $\partial_\mu\phi$ is rotation-free.