3.5 Yukawa potential.
Show that the Yukawa potential $V(r) = \mathrm{e}^{-mr}/(4\pi r)$ is the Fourier transform of $(\boldsymbol{k}^2+m^2)^{-1}$, cf. Eq. (3.37).

We introduce fist spherical coordinate and set $z = \cos\vartheta$

$$\int \mathrm{d}^3k \,\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{kr}}}{\boldsymbol{k}^2+m^2} = 2\pi \int_0^\infty \mathrm{d}k\, k^2 \int_{-1}^1 \mathrm{d}z\, \frac{\mathrm{e}^{\mathrm{i}krz}}{k^2+m^2} = 2\pi \int_0^\infty \frac{\mathrm{d}k\, k^2}{k^2+m^2} \frac{1}{-\mathrm{i}kr} \underbrace{\left(\mathrm{e}^{-\mathrm{i}kr} - \mathrm{e}^{\mathrm{i}kr}\right)}_{=-2\mathrm{i}\sin(kr)} \tag{49}$$

Since $k\sin(kr)$ is even, we can extend the integral, $\int_0^\infty \to \frac{1}{2}\int_{-\infty}^\infty$. Then

$$= \frac{2\pi}{r} \int_{-\infty}^\infty \mathrm{d}k\, k\, \frac{\sin(kr)}{k^2+m^2} = \frac{2\pi}{r} \int_{-\infty}^\infty \mathrm{d}k\, k\, \frac{\mathrm{e}^{\mathrm{i}kr} - \mathrm{e}^{-\mathrm{i}kr}}{2\mathrm{i}} = \frac{2\pi}{\mathrm{i}r} \int_{-\infty}^\infty \mathrm{d}k\, k\, \frac{\mathrm{e}^{\mathrm{i}kr}}{k^2+m^2}\,, \tag{50}$$

where we changed variable $k \to -k$ in the second term. Now we can use Cauchy's residue theorem, closing the path in the upper plane. Then we pick up from $k^2+m^2 = (k+\mathrm{i}m)(k-\mathrm{i}m)$ the single pole $k = +\mathrm{i}m$, ending up with

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{kr}}}{\boldsymbol{k}^2+m^2} = \frac{1}{(2\pi)^3} \frac{2\pi}{\mathrm{i}r}\, 2\pi\mathrm{i}\, \mathrm{i}m\, \frac{\mathrm{e}^{-mr}}{2\mathrm{i}m} = \frac{\mathrm{e}^{-mr}}{4\pi r} \tag{51}$$

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