3.8 Green functions.
Derive the conditions that the connected and the unconnected n-point Green functions are identical for $n = 2$,

$$G(x_1, x_2) = \mathcal{G}(x_1, x_2)\,.$$

Show that they differ in general for $n \geq 3$.

We start from the definition of the connected 2-point Green function and perform the differentiations,

$$\begin{aligned} G(x_1, x_2) &= \frac{1}{\mathrm{i}^2}\frac{\delta^2 \ln Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} = \frac{1}{\mathrm{i}^2}\frac{\delta}{\delta J(x_1)}\left[\frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x_2)}\right]\bigg|_{J=0} \\ &= \frac{1}{\mathrm{i}^2}\left[-\frac{1}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)} + \frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\right]\bigg|_{J=0}\,. \end{aligned} \tag{52}$$

The Green functions agree, $G(x_1, x_2) = \mathcal{G}(x_1, x_2)$, if $\delta Z[J]/\delta J(x)|_{J=0} = 0$ vanishes and $Z[0] = 1$. The latter holds, since we require $\langle 0|0\rangle = 1$. The former holds, because the vacuum should be empty, $\langle 0|\phi(x)|0\rangle = 0$ Taking further derivatives, the extra terms will not vanish anymore for $J = 0$, and thus in general $\mathcal{G}(x_1, \ldots, x_n) \neq G(x_1, \ldots, x_n)$ for $n > 2$.