3.9 ζ function regularisation.
a.) The function $f(t) = t/(\mathrm{e}^t - 1)$ is the generating function for the Bernoulli numbers B_n, i.e.

$$f(t) = \frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n . \tag{53}$$

Calculate the Bernoulli numbers B_n for $n \leq 3$.
b.) The Riemann ζ function can be defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{54}$$

for $s > 1$ and then analytically continued into the complex s plane. The Bernoulli numbers are connected to the Riemann ζ function with negative odd argument as

$$\zeta(1 - 2n) = -\frac{B_{2n}}{2n} . \tag{55}$$

This allows you to find with magical ease the sum needed in the Casimir energy (in 1+1 dimensions), $\sum_{n=1}^{\infty} n = \zeta(-1) = -B_2/2 = -1/12$. Less magically, show using

$$\frac{1}{a} \frac{a}{\mathrm{e}^a - 1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{B_n}{n!} a^n \tag{56}$$

that you can split the sum into a divergent and a finite part. The divergent term will cancel in the difference of the vacuum energy with and without plates, and the remaining finite term is determined by $B_2/2$.

a.) Work is easier, if even or odd Bernoulli numbers vanish above a minimal n. In order to check this, we want to divide $f(t)$ into its even and odd parts. This requires to replace e^t by terms like $\mathrm{e}^x \pm \mathrm{e}^{-x}$,

$$\frac{t}{\mathrm{e}^t - 1} + \frac{t}{2} = \frac{t}{2} \left(\frac{2}{\mathrm{e}^t - 1} + \frac{\mathrm{e}^{-t/2}}{\mathrm{e}^{-t/2}} \right) = \frac{t}{2} \left(\frac{\mathrm{e}^{t/2} + \mathrm{e}^{-t/2}}{\mathrm{e}^{t/2} - \mathrm{e}^{-t/2}} \right)$$

or

$$f(t) = \frac{t}{\mathrm{e}^t - 1} = \frac{t}{2} \coth \frac{t}{2} - \frac{t}{2}.$$

Since $t \coth t$ is an even function, all odd Bernoulli numbers except the first one vanish, $B_{2k+1} = 0$ for $k > 0$.
Then we calculate the first Bernoulli numbers, expanding the generating function in a power-series and comparing then the coefficients.

$$\begin{aligned} \frac{t}{\mathrm{e}^t - 1} &= \frac{1}{1 + \frac{1}{2}t + \frac{1}{3!}t^2 + \ldots} = 1 - \left[\frac{1}{2}t + \frac{1}{3!}t^2 + \ldots \right] + \left[\frac{1}{2}t + \ldots \right]^2 + \mathcal{O}(t^3) \\ &= 1 + \left(-\frac{1}{2} \right) t + \frac{1}{6} \left(\frac{1}{2!} \right) t^2 + \ldots . \end{aligned} \tag{57}$$

Thus $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_2 = \frac{1}{6}$.
An simpler way to derive higher Bernoulli numbers is the following: Setting $b^n = B_n$, the definition (53) becomes

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{b^n}{n!} t^n = \mathrm{e}^{bt} \tag{58}$$

or

$$t = \mathrm{e}^{(b+1)t} - \mathrm{e}^{bt} = (b+1)t - bt + \sum_{n=2}^{\infty} \frac{(b+1)^n - b^n}{n!} t^n \,. \tag{59}$$

The coefficients of t^n with $n \geq 2$ have to vanish, and thus $(b+1)^n = b^n = B_n$. For instance

$$b^2 = (b+1)^2 = b^2 + 2b + 1 \quad \Rightarrow \quad b = B_1 = -1/2 \,, \tag{60}$$

$$b^3 = (b+1)^3 = b^3 + \underbrace{3b^2}_{3B_2} + \underbrace{3b}_{-3/2} + 1 \quad \Rightarrow \quad b^2 = B_2 = 1/6 \,, \ldots \tag{61}$$

Hence this method requires as only input Pasquale's triangle for the binomial coefficients.

b.) We have to calculate the regularised sum $S = \lim_{a\to 0} \sum_{n=1}^{\infty} n\mathrm{e}^{-an}$. We repeat the first two steps of the lectures, but factor out this time e^a,

$$S = \sum_{n=1}^{\infty} n\mathrm{e}^{-an} = -\frac{\partial}{\partial a} \sum_{n=1}^{\infty} \mathrm{e}^{-an} = -\frac{\partial}{\partial a} \frac{1}{\mathrm{e}^a - 1}.$$

We recognise in the last term the generating function for the Bernoulli numbers, if we add a factor a. Thus

$$\frac{1}{a} \frac{a}{\mathrm{e}^a - 1} = \frac{1}{a} \sum_{n=0}^{\infty} \frac{B_n}{n!} a^n = \frac{1}{a} - \frac{1}{2} + \frac{a}{12} + 0 - \mathcal{O}(a^3).$$

Taking a derivative w.r.t. a and then the limit $a \to 0$ we obtain

$$S = -\frac{\partial}{\partial a} \sum_{n=1}^{\infty} \mathrm{e}^{-an} = \frac{1}{a^2} - \frac{1}{12} \,.$$

We have managed to separate the vacuum energy into a divergent term $\propto 1/a^2$ and a finite term. The divergent term will be cancelled by the same term in the "free" (without plates) term in the vacuum energy, the finite term remains and constitutes the physical result for the difference of the two vacuum energies.

As a bonus, let us show how one derives the relation (55). We start by generalising the previous definitions, introducing the Bernoulli polynomials $B_n(z)$ by

$$\frac{t\mathrm{e}^{tz}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} t^n \tag{62}$$

and the Hurwitz zeta function $\zeta_H(s, a)$ by

$$\zeta_H(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \,. \tag{63}$$

Next we rescale the integral representation of the Gamma function by $u = \lambda t$,

$$\Gamma(s) = \int_0^\infty \mathrm{d}u\, s^{z-1} \mathrm{e}^{-u} = \lambda^s \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-\lambda t}\,. \tag{64}$$

Then we set $\lambda = n + a$ and find

$$\frac{1}{(n+a)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-(n+a)t}\,. \tag{65}$$

Thus

$$\zeta_H(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \sum_{n=0}^\infty \mathrm{e}^{-(n+a)t} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \frac{\mathrm{e}^{-at}}{1-\mathrm{e}^{-t}}\,. \tag{66}$$

This defines an integral representation of the zeta function valid for $\Re(s) > 1$. The last factor becomes identical to the generating function (62), if we write $t^{s-1}\mathrm{e}^{-at} = t^{s-2} t \mathrm{e}^{-at}$.
Note that we are interested in $s = -k$, $k \in \mathbb{N}_0$. For these values, the Gamma function has simple poles and thus only the singular part of the integral can contribute. We extract this part splitting first the integral in two terms,

$$\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^1 \mathrm{d}t\, t^{s-2} \frac{(-t)\mathrm{e}^{-at}}{\mathrm{e}^{-t}-1} + \frac{1}{\Gamma(s)} \int_1^\infty \mathrm{d}t\, t^{s-2} \frac{(-t)\mathrm{e}^{-at}}{\mathrm{e}^{-t}-1}\,. \tag{67}$$

The integral in the second term is an entire function; therefore this term does not contribute to $\zeta_H(-k,a)$. Next we evaluate the first integral,

$$\int_0^1 \mathrm{d}t\, t^{s-2} \frac{(-t)\mathrm{e}^{-at}}{\mathrm{e}^{-t}-1} = \int_0^1 \mathrm{d}t\, t^{s-2} \sum_{n=0}^\infty \frac{B_n(a)}{n!}(-t)^n \tag{68}$$

$$= \sum_{n=0}^\infty \frac{B_n(a)}{n!} \frac{(-1)^n}{s-n+1} \tag{69}$$

Again only the singular term $n = k+1$ survives in $\zeta_H(-k,a)$,

$$\zeta_H(-k,a) = \lim_{\varepsilon\to 0} \frac{1}{\Gamma(-k+\varepsilon)} \frac{B_{k+1}(a)}{(k+1)!} \frac{(-1)^n}{\varepsilon} = -\frac{B_{k+1}(a)}{k+1}\,. \tag{70}$$

In the last step, we used an expansion of the Gamma function we derive below. Setting now $a = 0$, we obtain the desired relation (55) as special case.
Finally, we note that $\zeta_H(s,a)$ has a pole with residium $B_0(a) = 1$ for $s = -k = 1$. Extending (66) into the complete complex plane, one sees that this is the only pole of $\zeta_H(s,a)$.