4.5 Casimir effect.
Repeat the calculation of the Casimir effect for a scalar field in 3+1 dimensions [Hint: Convert the integration over transverse momenta into an integration over energy.]

Denoting with D^2 the area of the plates, with L their distance and setting $k_\perp = k_x k_y$ for the $d=2$ transverse dimensions, the energy between the plates is (per polarisation mode)

$$E = \left(\frac{D}{2\pi}\right)^2 \int \mathrm{d}k_x \int \mathrm{d}k_y \sum_{n=1}^{\infty} \frac{\omega}{2} = \frac{D^2}{4\pi} I \tag{87}$$

with

$$\omega = \sqrt{k_x^2 + k_y^2 + n^2 \frac{\pi^2}{L^2}}\,.$$

A simple and fast way is to follow the approach from the lectures, adding the cutoff function $\mathrm{e}^{-\varepsilon\omega}$. Introducing polar coordinates $\mathrm{d}k_x \mathrm{d}k_y = 2\pi \mathrm{d}k_\perp\, k_\perp$ and using then $\omega \mathrm{d}\omega = \mathrm{d}k_\perp\, k_\perp$, it follows

$$I = \int_0^\infty \mathrm{d}k_\perp\, k_\perp \sum_{n=1}^{\infty} \omega \mathrm{e}^{-\varepsilon\omega} = \sum_{n=1}^{\infty} \int_{nK}^{\infty} \mathrm{d}\omega\, \omega^2 \mathrm{e}^{-\varepsilon\omega} = \sum_{n=1}^{\infty} \int_{nK}^{\infty} \mathrm{d}\omega\, \frac{\partial^2}{\partial\varepsilon^2} \mathrm{e}^{-\varepsilon\omega} \tag{88}$$

where we set also $K = \pi/L$. Now we perform the ω integral, factor out $\mathrm{e}^{\varepsilon K}$ and perform then the sum,

$$I = \frac{\partial^2}{\partial\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\varepsilon} \mathrm{e}^{-\varepsilon n K} = \frac{\partial^2}{\partial\varepsilon^2} \frac{1}{\varepsilon} \mathrm{e}^{\varepsilon K} \sum_{n=0}^{\infty} \mathrm{e}^{-\varepsilon n K} = \frac{\partial^2}{\partial\varepsilon^2} \frac{1}{\varepsilon} \frac{1}{\mathrm{e}^{\varepsilon K} - 1} \tag{89}$$

We recognize in the last factor the generating function for the Bernoulli polynomials, and expand up to $\mathcal{O}(\varepsilon^0)$ terms in I using $B_4 = -1/30$,

$$I = \frac{\partial^2}{\partial\varepsilon^2} \frac{1}{\varepsilon} \frac{1}{\varepsilon K} \left[1 - \frac{\varepsilon K}{2} + \frac{(\varepsilon K)^2}{12} + 0 - \frac{(\varepsilon K)^4}{720} + \ldots\right] \tag{90}$$

$$= \frac{6}{\varepsilon^4 K} - \frac{2}{\varepsilon^3} + 0 - \frac{K^3}{360} + \ldots \tag{91}$$

The (transverse) energy density follows

$$\frac{E}{D^2} = \frac{I}{4\pi} = \frac{3L}{2\pi^2\varepsilon^4} - \frac{1}{2\pi\varepsilon^3} + \frac{\pi^2}{1440 L^3} + \ldots \tag{92}$$

The first term corresponds to the free vacuum energy and will drop out in the difference. The second term seems dangerous, but does not depend on the distance of the plates. Hence it does not contribute to the force, while the last one gives

$$F = -\frac{\partial}{\partial L}\left(\frac{\Delta E}{D^2}\right) = -\frac{\pi^2}{480 L^4}, \tag{93}$$

The corresponding result for the electromagnetic field should be multiplied by a factor two, accounting for the two polarization states of a photon.

As alternative, one can combine dimensional and zeta function regularization,

$$\begin{aligned} E &= \left(\frac{D}{2\pi}\right)^d \int \mathrm{d}k_\perp k_\perp^{d-1} \mathrm{d}^{d-1}\Omega \sum_{n=1}^{\infty} \frac{1}{2}\sqrt{k_\perp^2 + n^2\frac{\pi^2}{L^2}} && (94) \\ &= \left(\frac{D}{2\pi}\right)^d \frac{2\pi^{d/2}}{\Gamma(d/2)} \int \mathrm{d}k_\perp k_\perp^{d-1} \sum_{n=1}^{\infty} \frac{1}{2}\sqrt{k_\perp^2 + n^2\frac{\pi^2}{L^2}} && (95) \end{aligned}$$

Substituting $y = Lk_\perp/(n\pi)$ and using the definition of Euler's beta function,

$$B(x,y) = 2\int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{96}$$

the energy becomes

$$E = \left(\frac{D}{2\pi}\right)^d \frac{\pi^{d/2}}{2}\left(\frac{\pi}{L}\right)^{d+1} \frac{\Gamma(-d/2-1/2)}{\Gamma(-1/2)} \sum_{n=1}^{\infty} n^{d+1}. \tag{97}$$

Next we replace the divergent sum by the Riemann zeta function, $\zeta(s) \equiv \sum_{n=1}^\infty n^{-s}$. For even d, i.e. our case of interest, we can further replace the zeta function by Bernoulli polynomials and are done. For odd d, the Bernoulli polynomials are zero, while the Gamma function in the prefactor diverges. To proceed for general d, one can use the reflection formulae of both functions,

$$\zeta(-1-d) = 2^{-1-d}\pi^{-2-d}\sin\left[\frac{\pi}{2}(-1-d)\right]\Gamma(2+d)\zeta(2+d), \tag{98}$$

$$\Gamma\left(-\frac{d}{2}-\frac{1}{2}\right)\sin\left[\frac{\pi}{2}(-1-d)\right] = \frac{\pi}{\Gamma(d/2+3/2)} \tag{99}$$

to re-write the energy as

$$\frac{E}{D^d} = -\frac{1}{L^{d+1}}\frac{\Gamma(1+d/2)}{2^{d+2}\pi^{d/2+1}}\zeta(2+d). \tag{100}$$

For $d = 2$ transverse dimensions, one arrives with $\zeta(4) = \pi^4/90$ at

$$\frac{\Delta E}{D^2} = -\frac{\pi^2}{1440 L^3}\,. \tag{101}$$

We can check this result with the previous one obtained for 1+1 dimensions, or $d = 0$. Then our zeta-function regularized result reads

$$\frac{E}{D^0} = -\frac{1}{L}\frac{\Gamma(1)\zeta(2)}{4\pi} = -\frac{\pi}{24L}, \tag{102}$$

since $\Gamma(1) = 1$ and $\zeta(2) = \pi^2/6$.