5.4 Stress tensor for an ideal fluid.

a.) In the rest-frame of dust, $T^{00} = \rho$ and $u^\alpha = (1, \mathbf{0})$. Hence we can express the stress tensor as $T^{\alpha\beta} = \rho u^\alpha u^\beta$, an expression which is valid in an arbitrary frame. Writing $u^\alpha = (\gamma, \gamma \boldsymbol{v})$, we can identify $T^{00} = \gamma^2 \rho$ with the energy density, $T^{0i} = \gamma^2 \rho v^i$ with the energy/momentum density flux in direction i, and $T^{ij} = \gamma^2 \rho v^i v^j$ with the flow of the momentum density component i through the area with normal direction j.

For an ideal fluid in its rest-frame, the pressure is isotropic $P_{ij} = P\delta_{ij}$. This corresponds to $P_{ij} = -P\eta_{ij}$ and adds $-P$ to T^{00}. Compensating for this gives

$$T^{\alpha\beta} = (\rho + P) u^\alpha u^\beta - P\eta^{\alpha\beta} .$$

b.) In the nonrelativistic limit $v \ll c$, and thus $u^\alpha = (\gamma, \gamma\boldsymbol{v}) \to (1, \boldsymbol{v})$ and $\rho \gg P$. We set therefore first simply $P = 0$. We look first at the $\beta = 0$ component of $\partial_\alpha T^{\alpha\beta} = 0$, obtaining

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 . \tag{110}$$

This corresponds to the mass continuity equation and, because of $E = m$ for dust, at the same time to energy conservation. Next we consider the $\beta = j = 1, 2, 3$ components,

$$\partial_t (\rho v^j) + \partial_i (v^i v^j \rho) = 0 \tag{111}$$

or

$$\boldsymbol{v} \partial_t \rho + \rho \partial_t \boldsymbol{v} + \boldsymbol{v} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v} \rho = 0 . \tag{112}$$

Taking the continuity equation into account, we obtain the Euler equation for a force-free ideal fluid,

$$\rho \partial_t \boldsymbol{u} + \rho (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = 0 . \tag{113}$$

Including pressure, only $-P\eta^{\alpha\beta}$ affects the Euler equation (since $\rho \gg P$), contributing a $\boldsymbol{\nabla} P$ term to its RHS.

c.) From the Lagrangian $\mathscr{L} = -\rho_\Lambda$, we find $T^{\alpha\beta} = \rho_\Lambda \eta^{\alpha\beta}$. Comparing to the stress tensor of an ideal fluid at rest, we find $\rho = \rho_\Lambda$ and $P = -\rho_\Lambda$, or $w = -1$.