5.5 Stress tensor for the electromagnetic field.

i.) We start with the standard method, Noether's theorem, using the definition

$$T_\mu^{\ \nu} = \frac{\partial A_\sigma}{\partial x^\mu}\,\frac{\partial\mathscr{L}}{\partial(\partial A_\sigma/\partial x^\nu)} - \delta_\mu^\nu \mathscr{L}\,. \tag{114}$$

Since $\mathscr{L}$ depends only on the derivatives $A^\mu_{,\nu}$, we can use the following short-cut: We know already that

$$\delta\mathscr{L} = -\frac{1}{4}\,\delta(F_{\mu\nu}F^{\mu\nu}) = F^{\mu\nu}\,\delta(\partial_\nu A_\mu)\,. \tag{115}$$

Thus

$$\frac{\partial\mathscr{L}}{\partial(\partial A_\sigma/\partial x^\nu)} = F^{\sigma\nu} = -F^{\nu\sigma} \tag{116}$$

and

$$T_\mu^{\ \nu} = -\frac{\partial A_\sigma}{\partial x^\mu}\,F^{\nu\sigma} + \frac{1}{4}\,\delta_\mu^\nu F_{\sigma\tau}F^{\sigma\tau}\,. \tag{117}$$

Rearranging the indices, we have

$$T^{\mu\nu} = -\frac{\partial A^\sigma}{\partial x_\mu}\,F^\nu_{\ \sigma} + \frac{1}{4}\,\eta^{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\,. \tag{118}$$

This result in neither gauge invariant (it contains A^μ) nor symmetric. To symmetrize it, we should add

$$\frac{\partial A^\mu}{\partial x_\sigma}\,F^\nu_{\ \sigma} = \frac{\partial}{\partial x_\sigma}(A^\mu F^\nu_{\ \sigma})\,. \tag{119}$$

The last step is possible for a free electromagnetic field, $\partial_\sigma F^{\nu\sigma} = 0$. And since the term is a divergence and antisymmetric in $\nu\sigma$, we can add it without changing the conservation law for $T^{\mu\nu}$. The two terms combine to F, and we get

$$T^{\mu\nu} = -F^{\mu\sigma}\,F^\nu_{\ \sigma} + \frac{1}{4}\,\eta^{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\,. \tag{120}$$

This is gauge invariant and symmetric. Its trace is zero, $T_\mu^\mu = 0$. The 00 component is

$$T^{00} = -F^{0\sigma}\,F^0_{\ \sigma} + \frac{1}{4}\,\eta^{00}F_{\sigma\tau}F^{\sigma\tau}\,. \tag{121}$$

Using $F^{0k}\,F^0_{\ k} = -E^2$ and $F_{\sigma\tau}F^{\sigma\tau} = 2(B^2 - E^2)$, we obtain

$$T^{00} = -F^{0k}\,F^0_{\ k} + \frac{1}{2}\,(B^2 - E^2) = \frac{1}{2}(E^2 + B^2) \geq 0\,. \tag{122}$$

We identify $\rho = T^{00}$ and $P\delta^{ij} = T^{ij}$ comparing to the ideal fluid. Using then $T_\mu^\mu = \rho - 3P = 0$, the EoS $w = 1/3$ follows.

ii.) We use Newton's law

$$f_\mu = -\frac{\partial T^{\mu\nu}}{\partial x^\nu}\,,$$

where $T^{\mu\nu}$ is the stress tensor of the field acting with the forece density f_μ on external currents. Inserting the Lorentz force $f_\mu = F_{\mu\nu}j^\nu$ and Maxwell's equation $j^\nu = -\partial_\lambda F^{\nu\lambda}$ gives

$$f_\mu = F_{\mu\nu}j^\nu = -F_{\mu\nu}\,\frac{\partial F^{\nu\lambda}}{\partial x^\lambda}\,. \tag{123}$$

We use now the product rule to rewrite this as

$$-f_\mu = \frac{\partial}{\partial x^\lambda}\left(F_{\mu\nu}F^{\nu\lambda}\right) - F^{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda}\,. \tag{124}$$

We should rewrite the second term as a symmetric divergence. Starting from

$$F^{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda} = \frac{1}{2}\left(F_{\nu\lambda}\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + F_{\lambda\nu}\frac{\partial F_{\mu\lambda}}{\partial x^\nu}\right) \tag{125}$$

we have exchanged the indices λ and ν in the second term. Then we use first in both factors of the second term the antisymmetry of F,

$$= \frac{1}{2}\,F_{\nu\lambda}\left(\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu}\right) \tag{126}$$

then $\partial_\mu \tilde{F}^{\mu\nu} = 0$ and finally $\partial_\mu F^{\mu\nu} = 0$,

$$= \quad -\frac{1}{2}\,F_{\nu\lambda}\,\frac{\partial F_{\nu\lambda}}{\partial x^\mu} \tag{127}$$

$$= \quad -\frac{1}{4}\,\frac{\partial}{\partial x^\mu}\left(F_{\nu\lambda}F^{\nu\lambda}\right) = -\frac{1}{4}\,\delta^\lambda_\mu\,\frac{\partial}{\partial x^\lambda}\left(F_{\sigma\tau}F^{\sigma\tau}\right) \tag{128}$$

Combining, we get

$$-f_\mu = \frac{\partial}{\partial x^\lambda}\left(F_{\mu\nu}F^{\nu\lambda} + \frac{1}{4}\,\delta^\lambda_\mu\,F_{\sigma\tau}F^{\sigma\tau}\right) = \frac{\partial T^\lambda_\mu}{\partial x^\lambda}\,. \tag{129}$$

or

$$T_{\mu\nu} = -F_{\mu\lambda}F_\nu{}^\lambda + \frac{1}{4}\,\eta_{\mu\nu}\,F_{\sigma\tau}F^{\sigma\tau} \tag{130}$$

Considering the electromagnetic field as an external (i.e. fixed) field, the divergence of its energy-momentum tensor corresponds to the four-force density on charges.

iii.) We convert $\rho = T_{00} = (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$ into a tensor equation noting that this agrees with

$$\rho = T_{\alpha\beta}u^\alpha u^\beta$$

for an observer at rest, $u^\alpha = (1, 0)$. Our task is to massage the $\boldsymbol{E}^2 + \boldsymbol{B}^2$ term into the same form: First we use

$$\mathscr{L} = -\frac{1}{4}F^2 = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)\,,$$

obtaining

$$\boldsymbol{B}^2 = \boldsymbol{E}^2 + \frac{1}{2}F^2 = \boldsymbol{E}^2 + \frac{1}{2}F^2\eta_{\alpha\beta}u^\alpha u^\beta\,.$$

The energy density becomes

$$\rho = \boldsymbol{E}^2 + \frac{1}{4}F^2\eta_{\alpha\beta}u^\alpha u^\beta\,.$$

Next we work on the $\boldsymbol{E}^2$: We use $F_{0k} = E_k$ or $u^\alpha F_{\gamma\alpha} = E_\gamma$, obtaining

$$\boldsymbol{E}^2 = -E_\gamma E^\gamma = -u^\alpha F_{\gamma\alpha} F^\gamma{}_\beta u^\beta \,.$$

Combining the terms gives

$$\rho = (-F_{\gamma\alpha} F^\gamma{}_\beta + \frac{1}{4}\eta_{\alpha\beta} F^2) u^\alpha u^\beta$$

or

$$T_{\alpha\beta} = -F_{\alpha\gamma} F_\beta{}^\gamma + \frac{1}{4}\eta_{\alpha\beta} F^2 \,.$$