5.7 Linear Sigma model I.

Remark: This Lagrangian is called the "linear sigma model" which is both useful to understand the effect of spontanous symmetry breaking and, adding terms describing nucleons, as a model for low-energy nucleon-meson interactions. Moreover, comparing it to non-linear versions of the sigma model which share the same symmetries but look otherwise completely different shows that all models predict the same physics at low energies.

a.) First we note that $\mathscr{L} = \mathscr{L}^*$ implies that the fields are real. Following then the hint, we calculate

$$\Sigma\Sigma^\dagger = (\sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi})(\sigma - \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}) = \sigma^2 + (\boldsymbol{\tau}\cdot\boldsymbol{\pi})^2 = \sigma^2 + \boldsymbol{\pi}^2\,. \tag{138}$$

The last term is more precisely a matrix, $(\sigma^2 + \boldsymbol{\pi}^2)\mathbf{1}$, and thus we should take the trace to obtain a scalar. Thus the Lagrangian can be expressed as

$$\mathscr{L} = \frac{1}{4}\mathrm{tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) - \frac{1}{4}m^2\mathrm{tr}\left(\Sigma\Sigma^\dagger\right) - \frac{\lambda}{16}\mathrm{tr}\left(\Sigma\Sigma^\dagger\right)^2\,.$$

Next we use $\Sigma' = U\Sigma U^\dagger$, $\Sigma'^\dagger = U\Sigma^\dagger U^\dagger$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ or $\Sigma \propto \mathbf{1}$, to obtain

$$\mathrm{tr}\left(\Sigma\Sigma^\dagger\right) \to \mathrm{tr}\left(U\Sigma U^\dagger U\Sigma^\dagger U^\dagger\right) = \mathrm{tr}\left(\Sigma\Sigma^\dagger\right)\,.$$

Since U is constant, the derivatives in the kinetic term do not harm. Thus $\mathscr{L}$ is invariant under the (global) symmetry transformation (**??**).
To find the conserved currents, we consider infinitesimal transformations $U = \exp(\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2) \simeq 1 + \mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2$ or

$$\Sigma \to \Sigma' = U\Sigma U^\dagger \simeq \left(1 + \mathrm{i}\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right)(\sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi})\left(1 - \mathrm{i}\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}\right) \tag{139}$$

$$= \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} - \left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}, \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right] \tag{140}$$

Next we calculate the commutator,

$$\left[\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\tau}}{2}, \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right] = \mathrm{i}\boldsymbol{\alpha}\times\boldsymbol{\pi}\cdot\boldsymbol{\tau}$$

and find

$$\Sigma' = \sigma' + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi}' \simeq \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} - \mathrm{i}\boldsymbol{\tau}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\pi})$$

or

$$\delta\sigma = 0 \quad \text{and} \quad \delta\boldsymbol{\pi} = -\boldsymbol{\alpha}\times\boldsymbol{\pi}\,.$$

(As expected, the isoscalar σ is invariant, while $\boldsymbol{\pi}$ tranforms as a vector under rotation.) The conserved (isospin vector) current following from Noether's theorem is

$$-\boldsymbol{\alpha}\cdot\boldsymbol{V}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\boldsymbol{\pi})}\delta\boldsymbol{\pi} = -\partial^\mu\boldsymbol{\pi}\cdot(\boldsymbol{\alpha}\times\boldsymbol{\pi})\,.$$

or

$$\boldsymbol{V}^\mu = \partial^\mu\boldsymbol{\pi}\times\boldsymbol{\pi}\,.$$

b.) The Z_2 symmetry $\Sigma \to -\Sigma$ implies that one produces σ's and π^i always in pairs. Their scalar propagators are diagonal

$$\sigma \text{———} \sigma \qquad = \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

$$\pi^i \text{———} \pi^j \qquad = \frac{\mathrm{i}\, \delta^{ij}}{k^2 - m^2 + \mathrm{i}\varepsilon}$$

The structure of a $\Sigma^i \Sigma^k \cdots \Sigma^n$ vertex can be obtained calculating $\partial^n \mathscr{L}/(\partial\Sigma^i \partial\Sigma^k \cdots \partial\Sigma^n)$. This gives a factor $4!/4 = 6$ for four identical particles and $(2!)^2/2 = 2$ for two different pairs at the vertex. Thus a $ijkl$ vertex is $-2\mathrm{i}\lambda[\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}]$, i.e. $-6\mathrm{i}\lambda$ for four identical and $-2\mathrm{i}\lambda$ for two pairs at the vertex.