6.6 Inertial coordinates.

We proof it by construction. We choose new coordinates x'^a centered at P,

$$x'^a = x^a - x^a_P + \frac{1}{2}\Gamma^a_{bc}(x^b - x^b_P)(x^c - x^c_P). \tag{151}$$

Here Γ^a_{bc} are the connection coefficients in P calculated in the original coordinates x^a. We differentiate

$$\frac{\partial x'^a}{\partial x^d} = \delta^a_d + \Gamma^a_{db}(x^b - x^b_P))\,. \tag{152}$$

Hence $\partial x'^a/\partial x^d = \delta^a_d$ at the point P. Differentiating again,

$$\frac{\partial^2 x'^a}{\partial x^d \partial x^e} = \Gamma^a_{db}\delta^b_e = \Gamma^a_{de}\,. \tag{153}$$

Inserting these results into the transformation law of the connection coefficients,

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\,\frac{\partial x^e}{\partial x'^b}\,\frac{\partial x^f}{\partial x'^c}\,\Gamma^d_{ef} + \frac{\partial^2 x^d}{\partial x'^b \partial x'^c}\,\frac{\partial x'^a}{\partial x^d}\,. \tag{154}$$

where we swap in the second term derivatives of x and x',

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\,\frac{\partial x^f}{\partial x'^b}\,\frac{\partial x^g}{\partial x'^c}\,\Gamma^d{}_{fg} - \frac{\partial^2 x'^a}{\partial x^d \partial x^f}\,\frac{\partial x^d}{\partial x'^b}\,\frac{\partial x^f}{\partial x'^c} \tag{155}$$

gives

$$\Gamma'^a{}_{bc} = \delta^a_d\;\delta^f_b\;\delta^g_c\;\Gamma^d{}_{fg} - \Gamma^a{}_{df}\;\delta^d_b\;\delta^f_c = \Gamma^a{}_{bc} - \Gamma^a{}_{bc} \tag{156}$$

or

$$\Gamma'^a_{de}(P) = 0\,. \tag{157}$$

Thus we have found a coordinate system with vanishing connection coefficients at P. By a linear transformation (that does not affect ∂g_{ab}) we can bring finally g_{ab} into the form η_{ab}: As required by the equivalence principle, we can introduce in each point P of a space-time a free-falling coordinate system in which physics is described by the known physical laws in the absence of gravity.

Note that the introduction of Riemannian normal coordinates is in general only possible, if the connection is symmetric: Since the anti-symmetric part of the connection coefficients, the torsion, transforms as a tensor, it can not be eliminated by a coordinate change.