8.19 Linear Sigma model II.

a.) We introduce $N_L = P_L N = \frac{1}{2}(1-\gamma^5)N$ and $N_R = P_R N = \frac{1}{2}(1+\gamma^5)N$. Recalling then $\{\gamma^\mu, \gamma^5\} = 0$, $\gamma^5 P_L = -P_L$ and $\gamma^5 P_R = P_R$, we find

$$\bar{N}N = \bar{N}(P_L^2 + P_R^2)N = \bar{N}_L N_R + \bar{N}_R N_L \tag{187}$$
$$\bar{N}\partial\!\!\!/ N = \bar{N}_L \partial\!\!\!/ N_L + \bar{N}_R \partial\!\!\!/ N_R \tag{188}$$
$$\bar{N}\gamma^5 N = \bar{N}(P_L^2 + P_R^2)\gamma^5 N = \bar{N}_L N_R - \bar{N}_R N_L \tag{189}$$

The Yukawa interactions become

$$\bar{N}(\sigma + \mathrm{i}\gamma^5 \boldsymbol{\tau\pi})N = \bar{N}_L \sigma N_R + \bar{N}_R \sigma N_L + \bar{N}_L \mathrm{i}\boldsymbol{\tau\pi} N_R - \bar{N}_R \mathrm{i}\boldsymbol{\tau\pi} N_L = \bar{N}_L \Sigma N_R + \bar{N}_R \Sigma^\dagger N_L \tag{190}$$

We have now all ingredients necessary to check the invariance of the new terms. The L kinetic term transforms as

$$\bar{N}'_L \partial\!\!\!/ N'_L = \bar{N}_L U^\dagger \partial\!\!\!/ U N_L = \bar{N}_L \partial\!\!\!/ N_L \tag{191}$$

and is thus invariant; similiarly for the R term. For the Yukawa terms we obtain

$$\bar{N}'_L \Sigma' N'_R = \bar{N}_L U^\dagger U \Sigma U^\dagger U N_R = \bar{N}_L \Sigma N_R \tag{192}$$
$$\bar{N}'_R \Sigma^{\dagger\prime} N'_L = \bar{N}_R U^\dagger U \Sigma^\dagger U^\dagger U N_L = \bar{N}_R \Sigma^\dagger N_L\,. \tag{193}$$

Thus the new interactions terms are invariant too.
We find the new contribution to the Noether current by calculating the change δN under an infinitesimal transformation,

$$\delta N = \mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2N$$

Thus

$$-\boldsymbol{\alpha}\cdot\boldsymbol{V}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu N)}\delta N = -\bar{N}\gamma^\mu \boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2N \tag{194}$$

or

$$\boldsymbol{V}^\mu = \bar{N}\gamma^\mu \frac{\boldsymbol{\tau}}{2} N\,.$$

b) Show that the Lagrangian $\mathscr{L}$ is invariant under axial isospin transformations

$$N \to N' = \exp\left(\mathrm{i}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma^5\right)N \quad \text{and} \quad \Sigma \to \Sigma' = V^\dagger \Sigma V^\dagger\,, \tag{195}$$

where $V = \exp(\mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\tau}/2)$, $\boldsymbol{\beta} = (\beta^1, \beta^2, \beta^3)$ are real parameters and $\boldsymbol{\tau} = (\tau^1, \tau^2, \tau^3)$ are again the Pauli matrices. Find the corresponding conserved Noether current.

b.) Life is easier on the linear level,

$$N_L \to N'_L = P_L N' \simeq P_L \left(1 + \mathrm{i}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\gamma^5\right)N = \left(1 - \mathrm{i}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right)P_L N = V^\dagger N_L \tag{196}$$

and $N_R \to N'_R = V N_R$. We know that the kinetic term of the nucleons is invariant, while the Yukawa terms transform as

$$\bar{N}'_L \Sigma' N'_R = \bar{N}_L V V^\dagger \Sigma V^\dagger V N_R = \bar{N}_L \Sigma N_R \tag{197}$$

$$\bar{N}'_R \Sigma^{\dagger\prime} N'_L = \bar{N}_R V^\dagger V \Sigma^\dagger V V^\dagger N_L = \bar{N}_R \Sigma^\dagger N_L \,, \tag{198}$$

i.e. they are invariant too.
Next we use $\Sigma' = V^\dagger \Sigma V^\dagger$, $\Sigma'^\dagger = V \Sigma^\dagger V$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ or $\Sigma \propto \mathbf{1}$, to obtain

$$\mathrm{tr}\left(\Sigma\Sigma^\dagger\right) \to \mathrm{tr}\left(V^\dagger \Sigma V^\dagger V \Sigma^\dagger V\right) = \mathrm{tr}\left(\Sigma\Sigma^\dagger\right) .$$

To find the conserved current, we consider infinitesimal transformations

$$\Sigma \to \Sigma' = V^\dagger \Sigma V^\dagger \simeq \left(1 - \mathrm{i}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right)(\sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi})\left(1 - \mathrm{i}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}\right) \tag{199}$$

$$= \sigma + \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{\pi} - \mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\tau}\sigma + \boldsymbol{\beta}\cdot\boldsymbol{\pi} \,. \tag{200}$$

where we used $\{\tau^i, \tau^j\} = 2\delta^{ij}$ and

$$\left\{\frac{\boldsymbol{\beta}\cdot\boldsymbol{\tau}}{2}, \boldsymbol{\tau}\cdot\boldsymbol{\pi}\right\} = \boldsymbol{\beta}\cdot\boldsymbol{\pi} \,.$$

Thus $\delta\sigma = \boldsymbol{\beta}\cdot\boldsymbol{\pi}$ and $\delta\boldsymbol{\pi} = -\sigma\boldsymbol{\beta}$. Combined with $\delta N = \mathrm{i}\boldsymbol{\beta}\cdot\boldsymbol{\tau}\gamma^5 N/2$, the current is

$$-\boldsymbol{\beta}\cdot\boldsymbol{A}^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu N)}\delta N + \frac{\partial\mathscr{L}}{\partial(\partial_\mu \sigma)}\delta\sigma + \frac{\partial\mathscr{L}}{\partial(\partial_\mu \boldsymbol{\pi})}\delta\boldsymbol{\pi} = -\bar{N}\gamma^\mu\boldsymbol{\alpha}\cdot\boldsymbol{\tau}/2\gamma^5 N + \partial^\mu\sigma(\boldsymbol{\beta}\cdot\boldsymbol{\pi}) - \sigma\partial^\mu\boldsymbol{\pi}\cdot\boldsymbol{\beta} \,. \tag{201}$$

or

$$\boldsymbol{A}^\mu = \bar{N}\gamma^\mu\gamma^5\frac{\boldsymbol{\tau}}{2}N - (\boldsymbol{\pi}\partial^\mu\sigma - \sigma\partial^\mu\boldsymbol{\pi}) \,.$$

Remark: One can combine vector and axial transformations as follows:

$$N_L \to N'_L = L N_L \qquad N_R \to N'_R = R N_R \quad \text{and} \quad \Sigma \to \Sigma' = V^\dagger \Sigma V^\dagger \,, \tag{202}$$

where $L = \exp(\mathrm{i}\boldsymbol{\gamma}\cdot\boldsymbol{\tau}/2)$, $R = \exp(\mathrm{i}\boldsymbol{\delta}\cdot\boldsymbol{\tau}/2)$, and $\gamma = \delta = \alpha$ for vector transformations, $-\gamma = \delta = \beta$ for axial transformations. Calculating then the Noether currents, one finds

$$\boldsymbol{V}^\mu = \boldsymbol{R}^\mu + \boldsymbol{L}^\mu \quad \text{and} \quad \boldsymbol{A}^\mu = \boldsymbol{R}^\mu - \boldsymbol{L}^\mu \,. \tag{203}$$

c) Consider again the case of symmetry breaking, $m^2 \to -\mu^2$: What happens with the nucleons? What happens with the symmetries?

c.) Inserting $\sigma = v + \tilde{\sigma}$ into the Yukawa term, we obtain a mass term $m\bar{N}N$ with $m = gv$ for the nucleons.
The symmetry is broken from $\mathrm{SU(2)_L \otimes SU(2)_R}$ (or $\mathrm{SU(2)_V \otimes SU(2)_A}$) to $\mathrm{SU(2)_V}$. Equivalently, SO(4) is broken to SO(3).