

8.4 g_s -factor of the electron.

Squaring the Dirac equation and using $i\sigma^{\mu\nu}D_\mu D_\nu = \frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu] = \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$ gives

$$(i\not{D} + m)(i\not{D} - m)\psi = -(\not{D}\not{D} + m^2)\psi = -\left(D_\mu D^\mu - \frac{e}{2}\sigma_{\mu\nu}F^{\mu\nu} + m^2\right)\psi = 0. \quad (182)$$

Comparing to the Klein-Gordon equation, we see that the spin of the electron leads to the additional interaction term $\sigma_{\mu\nu}F^{\mu\nu}$. Consider now a homogeneous magnetic field $\mathbf{B} = B\mathbf{e}_z$ or $F_{12} = B$. We choose the gauge $A_\mu = (0, -\frac{1}{2}Bx^2, \frac{1}{2}Bx^1, 0)$. Moreover we consider weak fields, neglecting quadratic terms in A . Then

$$D_i^2 \simeq \nabla^2 - ie(\partial_i A_i + A_i \partial_i) = \nabla^2 - 2ieA_i \partial_i = \nabla^2 - 2\frac{ie}{2}B(x^1 \partial_2 - x^2 \partial_1) = \nabla^2 - \mathbf{B}\mathbf{L}.$$

(Since $D_i^2 \psi$, we applied the product rule in the second step.)

In the Dirac basis, a non-relativistic electron is $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \simeq \begin{pmatrix} \phi \\ 0 \end{pmatrix}$. With $\sigma^{ij} = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ it is

$$\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} = \frac{e}{2}\sigma^3(F_{12} - F_{21}) = e\sigma^3 B = 2e\mathbf{s}\mathbf{B}.$$

Combined we find $H_{\text{int}} = (\mathbf{L} + 2\mathbf{s})e\mathbf{B}$, i.e. the $g_s = 2$ factor of the electron

This was the first crucial test that the Dirac equation describes correctly electrons.