8.8 Expectation values in charge-conjugated states.

Remark: The definition makes sense as a comparison with our result $\rho = \psi^\dagger(x)(\mathrm{i}\partial_t)\psi(x)$ for the energy density shows.

a.)

$$\langle O\rangle_c = \langle\psi_c|\, O\,|\psi_c\rangle = \int \mathrm{d}^3x(\mathrm{i}\gamma^2\psi^*)^\dagger O\,\mathrm{i}\gamma^2\psi^* = \int \mathrm{d}^3x\psi^{*\dagger}\underbrace{\gamma^{2\dagger}}_{=-\gamma^2} O\,\gamma^2\psi^* \tag{183}$$

$$= -\left(\int \mathrm{d}^3x\psi^\dagger(\gamma^2 O\gamma^2)^*\psi\right)^* = -\langle\psi|\,\gamma^2 O^*\gamma^2\,|\psi\rangle^* \tag{184}$$

where we used $\gamma^{2\dagger} = -\gamma^2$ and $\gamma^{2*} = -\gamma^2$.

b.) In the coordinate representation $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}$ and

$$\langle\boldsymbol{p}\rangle_c = -\langle\psi|\,\gamma^2(+\mathrm{i}\boldsymbol{\nabla})\gamma^2\,|\psi\rangle^* = -\langle\psi|\,(-\mathrm{i}\boldsymbol{\nabla})\,|\psi\rangle^* \tag{185}$$

$$= -\langle\psi|\,(-\mathrm{i}\boldsymbol{\nabla})\,|\psi\rangle = -\langle\boldsymbol{p}\rangle \tag{186}$$

where we used going from (185) to (186) that the expectation value of $\boldsymbol{p}$ is real.

c.) Direct calculation of $\gamma^2\boldsymbol{\Sigma}^*\gamma^2$ gives $\mathrm{diag}(\boldsymbol{\sigma}^{*T}, \boldsymbol{\sigma}^{*T})$ and thus $\langle\boldsymbol{\Sigma}\rangle_c = -\langle\boldsymbol{\Sigma}\rangle$.