9.1 Optical theorem.

a. The Feynman amplitude for the decay is $\mathrm{i}\mathcal{A} = -\mathrm{i}g$. Thus the angular integration gives simply 4π. In the rest frame of the decaying particle, $M^2 = 4(m^2 + p_{\rm cms}^2)$. Combined we find

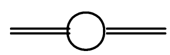
$$\Gamma = \frac{g^2}{2M}\,\frac{1}{32\pi^2}\sqrt{1-\frac{4m^2}{M^2}}\,4\pi = \frac{g^2}{16\pi M}\,\sqrt{1-\frac{4m^2}{M^2}}\,.$$

b. The scattering amplitude consists of the s and the u channel exchange of the heavy scalar Φ,

$$\mathrm{i}\mathcal{A} = (-\mathrm{i}g)^2\left[\frac{\mathrm{i}}{s-M^2+\mathrm{i}\varepsilon}+\frac{\mathrm{i}}{u-M^2+\mathrm{i}\varepsilon}\right]$$

with $s=(p_1+p_2)^2$ and $u=(p_2'-p_1)^2$. The second denominator never vanishes, while the first is zero for $s=M^2$, i.e. when the virtual scalar Φ is created on-shell. If we do not take the finite life-time of the heavy particle into account, it can travel (as a real particle) for infinite time, leading to an infinite range of the interaction.

c. The Feynman diagram is

.

The Feynman rules give

$$\mathrm{i}\Sigma = (-\mathrm{i}g)^2\int\frac{\mathrm{d}^4q}{(2\pi)^4}\,\frac{\mathrm{i}}{q^2-m^2+\mathrm{i}\varepsilon}\,\frac{\mathrm{i}}{(q-p)^2-m^2+\mathrm{i}\varepsilon}$$

Setting $p=(M,0)$ and $E_q=+\sqrt{q^2+m^2}$, we find as poles of the integrand $q^0=E_q-\mathrm{i}\varepsilon$, $q^0=-E_q+\mathrm{i}\varepsilon$, $q^0=M+E_q-\mathrm{i}\varepsilon$, and $q^0=M-E_q+\mathrm{i}\varepsilon$. We can choose the integration contour either in the upper or lower half-plane. Choosing the lower one, we pick up the two residues at $q^0=E_q-\mathrm{i}\varepsilon$ and $q^0=M+E_q-\mathrm{i}\varepsilon$. Hence we obtain

$$\Sigma = -g^2\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2ME_q}\left(\frac{1}{M-2E_q+\mathrm{i}\varepsilon}+\frac{1}{M+2E_q-\mathrm{i}\varepsilon}\right)$$

The second denominator never vanishes and thus gives no contribution to the imaginary part. For the first one, we obtain using the given identity

$$\mathrm{Im}\Sigma = g^2\pi\int\frac{\mathrm{d}^3q}{(2\pi)^3}\,\frac{1}{2ME_q}\delta(M-2E_q)$$

As $E_q=+\sqrt{q^2+m^2}\geq m$, the argument of the delta function is never zero for $M\leq 2m$ and the imaginary part of the amplitude vanishes thus. For $M>2m$, we can perform the integral,

$$\mathrm{Im}\Sigma = \frac{g^2}{16\pi}\,\sqrt{1-\frac{4m^2}{M^2}}$$

Thus we confirmed the relation $M\Gamma=\mathrm{Im}\Sigma$.