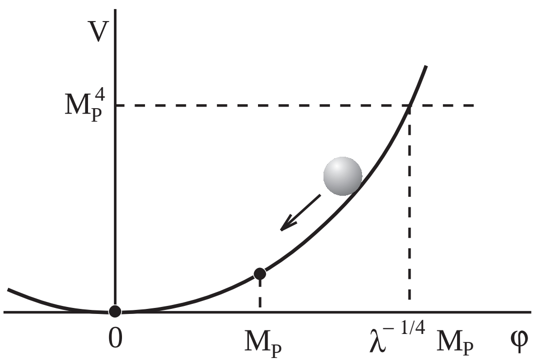


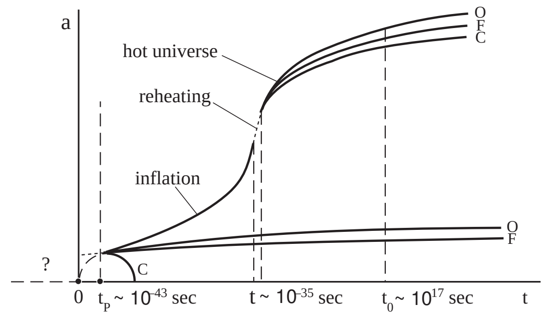


Figure 12.1: Left: A slowly rolling scalar field as model for inflation. Right: The evolution of the scale factor R including an inflationary phase in the early universe.

12.2 Structure formation

12.2.1 Overview and data

- structure formation operates via gravitational instability, but needs as starting point a seed of primordial fluctuations (generated in inflation)
- growth of structure is inhibited by many factors, e.g. pressure.
  The distance travelled by a freely falling particle is $R \sim gt^2/2$ with $g = GM/R^2$; or $t \sim \sqrt{R^3/GM} \sim \sqrt{1/G\rho}$. Thus $\tau_{\rm ff} \sim 1/\sqrt{G\rho}$.
  Pressure can balance gravity, if $\tau_{\rm ff} \gtrsim \lambda/v_s$. This defines a critical length ("Jeans length")
  $$\lambda \sim \frac{v_s}{\sqrt{G\rho}},$$
  below which pressure can counteract density perturbations (resulting in acoustic oscillations), above the density perturbation grows. Shows already that structure formation is sensitive to E.o.S. (compare e.g. radiation with $v_s^2 = 1/3$ with baryonic matter $v_s^2 = 5T/(3m)$).
- if growth of perturbation leads to $\Omega \geq 1$ in a region, the region decouples from the Hubble expansion and collapses.
- Assume $\rho = \rho_m + \rho_\gamma$. If perturbations in ρ are adiabatic, i.e. the entropy per baryon is conserved, $\delta(\rho_m/s) = 0$ or $\delta \ln(\rho_m/T^3) = 0$, then $\delta \ln \rho_m - 3\delta \ln T = 0$ or
  $$\frac{\delta\rho_m}{\rho_m} = 3\,\frac{\delta T}{T}$$
  [Another possibility would be $\delta\rho = 0$ or $\delta\rho_m = -\delta\rho_\gamma = -4aT^3\delta T = -4\rho_\gamma\delta T/T$ and $4\delta T/T = -\delta\rho_m/\rho_\gamma = -(\rho_m/\rho_\gamma)\,(\delta\rho_m/\rho_m)$. In the radiation epoch $\rho_m/\rho_\gamma \ll 1$ and temperature fluctuations are suppressed.]

⇒ temperature fluctuation in CMB at $z \approx 1100$ and matter fluctuation today $0 \leq z \lesssim 5$ have the same origin, if primordial fluctuations are adiabatic.

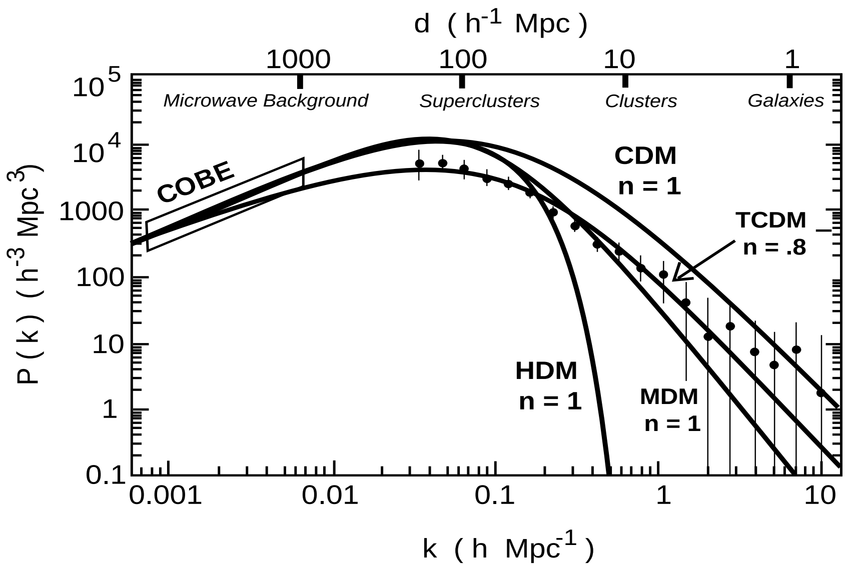


Figure 12.2: Comparison of the predicted power spectrum normalized to COBE data in several models popular around '95 with observations: HDM ($\Omega_\nu = 1$), CDM ($\Omega_m = 1$) and MDM ($\Omega_m = 0.8$, $\Omega_\nu = 0.2$).

- Basics of structure formation:
  assume initial fluctuations and examine how they are transformed by gravitational instability, interactions and free-streaming of different particle species

- comparison with observations via i) power-spectrum $P(k) = |\delta_k|^2$, where

$$\delta_k \propto \int d^3x\, e^{-ikx}\delta(x) \propto k^{n_s} \quad \text{with} \quad \delta(x) \equiv \frac{\rho(x) - \bar{\rho}}{\bar{\rho}}$$

  or ii) correlation function $\int d^3x\, n(x)n(x + x_0)$ or normalized

$$\xi(x_0) \equiv \frac{1/V \int d^3x\, n(x)n(x + x_0)}{(1/V \int d^3x\, n(x)n(x + x_0))^2} - 1$$

  The correlation function is the Fourier-transform of the power spectrum.
  Typical $\xi \approx (r/r_0)^\gamma$ with $\gamma \sim 1.8$ for $0.1 \lesssim r \lesssim 10$Mpc.

- An example of the status 1995 is shown in Fig. 12.2. Field is driven by tremendous growth of data:
  galaxy catalogues: Hubble '32: 1250, Abell '58: 2712 cluster, 2dF : 250.000, SDSS (-'08): 10^6.
  CMB experiments: '65 detection, COBE '92: anisotropies, towards '00: first peak, ...
  N-body simulations: Peeble '70: N=100, Efstahiou, Eastwood '81: 20.000, 2005: Virgo: Millenium 10^6

Jeans instability for a non-expanding fluid The three basic equations describing a perfect fluid in a Newtonian picture are the conservation equation for mass, for momentum, and the Poisson equation,

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0\,, \tag{12.44a}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \nabla) \vec{v} = \vec{F} - \nabla P\,, \tag{12.44b}$$

$$\Delta \phi = 4\pi G \rho\,. \tag{12.44c}$$

The LHS of (12.44b), the Euler equation, measures the velocity change $\mathrm{d}v/\mathrm{d}t$ of a fluid element, summing up the change at a fixed coordinate, $\partial_t v$, and the change due to the movement of the element, while the RHS consists of an external force $\vec{F}$ and a force due to a pressure gradient ∇P. [Remember that we derived the first two equations with $F = P = 0$ already from the energy-momentum conservation for dust; thus they correspond to $\partial_a T^{ab} = F^b_{\mathrm{ext}}$ for a fluid with non-zero pressure.] The Poisson equation connects the mass density ρ with the gravitational potential ϕ.

We consider now small perturbations x_1 around a static background, $\rho_0 = \mathrm{const.}$, $P_0 = \mathrm{const.}$ and $v_0 = 0$. Restricting us to small perturbations, $x_1 \ll x_0$, allows us to neglect all quantities quadratic in the perturbations. Sound waves propagate in most circumstances adiabatically, i.e. without production of entropy, $\mathrm{d}S = 0$. Thus changes in P and ρ are connected via

$$P = P_0 + \left(\frac{\partial P}{\partial \rho}\right)_S \mathrm{d}\rho + \left(\frac{\partial P}{\partial S}\right)_P \mathrm{d}S = P_0 + c_s^2 \mathrm{d}\rho\,. \tag{12.45}$$

Inserting $x = x_0 + x_1$ into the fluid equations and neglecting quadratic term gives

$$\partial_t \rho_1 + \rho_0 \nabla \cdot \vec{v}_1 = 0\,, \tag{12.46a}$$

$$\Delta \phi_1 = 4\pi G \rho_1\,, \tag{12.46b}$$

$$\partial_t \vec{v}_1 + \frac{c_s^2}{\rho_0} \nabla \rho_1 + \nabla \phi_1 = 0\,, \tag{12.46c}$$

where we used also $\partial_x P = \frac{\partial \rho_1}{\partial x} \frac{\partial}{\partial \rho_1} P_1 = \frac{\partial \rho_1}{\partial x} c_s^2$. These three equations can be combined into one second-order differential equation for ρ_1. We multiply Eq. (12.46c) by ρ_0 and apply ∇ on it,

$$c_s^2 \Delta \rho_1 = -\rho_0 (\partial_t \nabla \cdot \vec{v}_1 + \Delta \phi_1)\,. \tag{12.47}$$

Then we insert (12.46b) for $\Delta \phi_1$ and (12.46a) for $\nabla \cdot \vec{v}_1$, and obtain a linear, inhomogeneous wave equation

$$\partial_t^2 \rho_1 - \underbrace{c_s^2 \Delta \rho_1}_{\text{pressure}} = \underbrace{4\pi G \rho_1 \rho_0}_{\text{grav.force}} \tag{12.48}$$

for the density perturbation ρ_1. The dispersion relation of plane waves $\exp(-i(\omega t - kx))$,

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0\,, \tag{12.49}$$

confirms that $c_s = (\partial P/\partial \rho_1)^{1/2}$ is the sound speed.

For $k > k_J$, where the Jeans wave number is

$$k_J = \left(\frac{4\pi G \rho_0}{v_s^2}\right)^{1/2} \tag{12.50}$$

there are acoustic oscillations, for $k < k_J$ the modes are exponentially growing (or decaying).

The total mass M_J contained within a sphere of radius $\lambda_J/2 = \pi/k_J$ is

$$M_J = \frac{4\pi}{3}\left(\frac{\pi}{k_J}\right)^3 \rho_0 = \frac{\pi^{5/2}}{6}\frac{v_s^3}{G^{3/2}\rho_0^{1/2}} \tag{12.51}$$

The Jeans mass is constant in an expanding universe, since the wave-number $k_J \propto 1/R$ and $\rho_0 \propto 1/R^3$.

Jeans mass of baryons Consider[2] mixture of radiation and non-relativistic nucleons after e^+e^- anihilations, i.e. $T \approx 0.5$ MeV. With $\rho = \rho_m + \rho_\gamma$ and $P \approx P_\gamma = \rho_\gamma/3$, we have

$$v_s = \left(\frac{\partial\rho}{\partial P}\right)_S^{1/2} = \frac{1}{\sqrt{3}}\left(1 + \frac{\partial\rho_m}{\partial\rho_\gamma}_S\right)^{-1/2} = \frac{1}{\sqrt{3}}\left(1 + \frac{3\rho_m}{4\rho_\gamma}\right)^{-1/2} \tag{12.52}$$

where we used $\frac{\delta\rho_m}{\rho_m} = 3\,\frac{\delta T}{T} = \frac{3\delta\rho_\gamma}{4\rho_\gamma}$.

For $t \ll t_{\rm eq}$, the adiabatic sound speed is close to $v_s = 1/\sqrt{3}$, while $v_s = 0.76/\sqrt{3}$ for $t = t_{\rm eq}$. The Jeans mass of baryons is close to the horizon size until recombination. Then v_s drops to the value for a mono-atomic gas, $v_s^2 = \frac{5T_b}{3m}$, where $m \sim m_H \sim 1$ GeV.

Let us compare the Jeans mass just before and after recombination,

$$M_J(z_{\rm eq,>}) = \frac{\pi^{5/2}}{6}\frac{v_s^3}{G^{3/2}\rho^{1/2}} \sim 10^{15}\left(\Omega h^2\right)^{-2} M_\odot \tag{12.53}$$

and

$$M_J(z_{\rm eq,<}) \sim 10^5\left(\Omega h^2\right)^{-1/2} M_\odot \tag{12.54}$$

The Jeans mass of baryons does not coincide with the oberved mass of galaxies, neither fits the corresponding length scale the break in the power spectrum around $k \approx 0.04h/$ Mpc.

12.2.2 Damping scales

Collisional or Silk damping Consider which fluctuations are damped by dissipative processes, e.g. by Thomson scattering. (The mean free path of photons is always much larger than the one of electrons, $l_\gamma = (n_e\sigma_T)^{-1} \gg l_e = (n_\gamma\sigma_T)^{-1}$, since $n_\gamma \sim 10^{10} n_e$. Thus photon diffusion is much more important then electron diffusion.)

A sound wave with wavelength λ can be damped if the diffusion time $\tau_{\rm diff}$ is smaller than the Hubble time t_H. Estimate $\tau_{\rm diff}$ by a random-walk with $N = \lambda^2/l_{\rm int}^2$ steps, each with size $l_{\rm int} = 1/n_e\sigma_{th}$,

$$\tau_{\rm diff} = N l_{\rm int} = \lambda^2/l_{\rm int} < t_H\,. \tag{12.55}$$

Thus the damping scale is $\lambda_D = (l_{\rm int} l_H)^{1/2}$. If there would be a baryon-dominated epoch, then $n_e \propto \rho$ and $\rho \propto 1/t^2$, hence $l_{\rm int} l_H \propto \rho^{-1-1/2}$ and $\lambda_D \propto \rho^{-3/4}$. Finally, the corresponding mass scale is $M_D \propto \lambda^3\rho \propto \rho^{-9/4}\rho \propto \rho^{-5/4}$. Numerically,

$$M_D = 10^{12}(\Omega h^2)^{-5/4} M_\odot \tag{12.56}$$

[2]Just to round up the discussion

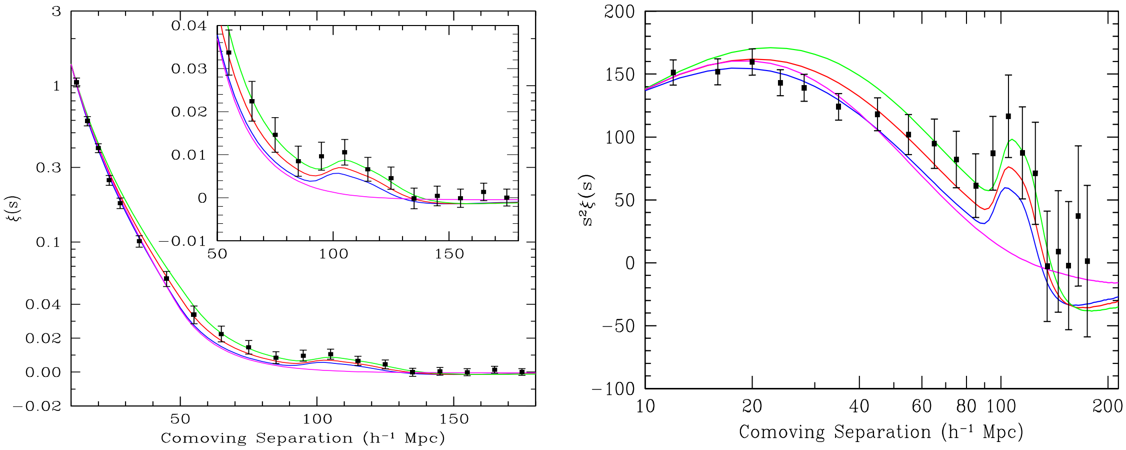


Figure 12.3: Acoustic baryon oscillation in the correlation function of galaxies with large redshift of the SDSS, astro-ph/0501171.

and the corresponding length scale is (taking into account $\Omega_b \approx 0.04 < \Omega_m \approx 0.27$ and $h \approx 0.7$)

$$\lambda_D = 3.5(\Omega_m/\Omega_b)^{1/2}(\Omega h^2)^{-3/4}\,\mathrm{Mpc} \approx 40\,\mathrm{Mpc}\,. \tag{12.57}$$

Thus the Silk scale λ_D has the right numerical value to explain the break in the power spectrum at $k \approx 0.04h/\,\mathrm{Mpc}$. Fluctuations are damped on scales $\lambda \gtrsim \lambda_D$, the stronger the larger Ω_b. Since $M_D \gg M_J$, acoustic oscillation should be visible for $k \gtrsim k_D$ in the power spectrum of galaxies. First evidence was found last years, cf. Fig. 12.3.

12.2.3 Grow of perturbations in an expanding Universe:

We restrict ourselves to the simplest case: perturbations in a pressurless, expanding medum. Starting from a homogenous universe, we add matter inside a sphere of radius R, $\bar{\rho} \to \bar{\rho}(1+\delta)$. Then the acceleration on the surface of this sphere is

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho}\delta\,. \tag{12.58}$$

The time evolution of the mass density is

$$\rho(t) = \bar{\rho}(t)[1+\delta(t)] = \bar{\rho}_0/a^3(t)[1+\delta(t)] \tag{12.59}$$

and thus mass conservation

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta]R^3 = \text{const.} \tag{12.60}$$

implies

$$R(t) \propto a(t)[1+\delta]^{-1/3}\,. \tag{12.61}$$

Expanding for $\delta \ll 1$ and differentiating gives

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} - \frac{1}{3}\ddot{\delta} \tag{12.62}$$

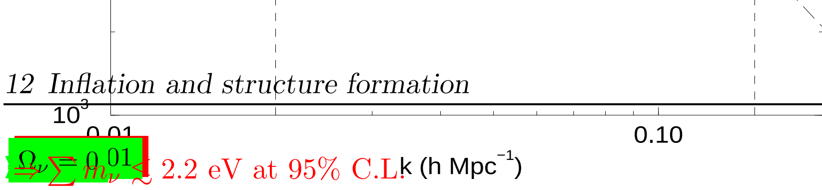


Figure 12.4: Neutrino mass limits from the 2dF galaxy survey: For $\Omega_\nu \gtrsim 0.05$ there is too less power on scales smaller than (or since normalization is arbitrary) slope too steep).

Combined with (12.58) it gives

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho\delta = 0} \tag{12.63}$$

For a matter-dominated universe, $H = 2/(3t)$ and $\rho = 1/(6\pi G t^2)$. Inserting the trial solution $\delta \propto t^\alpha$ gives

$$\alpha(\alpha-1)t^{\alpha-2} + \frac{4}{3}\alpha t^{\alpha-2} - \frac{2}{3}t^{\alpha-2} = 0 \tag{12.64}$$

or

$$\alpha^2 + \frac{1}{3}\alpha - \frac{2}{3} = 0 \tag{12.65}$$

or $\alpha = -1$ and $2/3$. Thus the general solution $\delta(t) = At^{-1} + Bt^{2/3}$ consists of a decaying mode $\delta \propto 1/t$ and a mode growing like $\delta \propto t^{2/3} \propto R$.
During the radiation-dominated epoch, with $\delta_\gamma = 0$, one can neglect the term $4\pi G\rho\delta$. With $H = 1/(2t)$

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} = 0 \tag{12.66}$$

with solution $\delta(t) = \delta(t_i)[1 + a\ln(t/t_i)]$. Thus perturbations do not grow until $z_{\rm eq}$.

Summary of different effects

- on sub-horizon scales our Newtonian anlysis applies. During the radiaton-dominated epoch, perturbations do not grow. During the matter-dominated epoch, perturbations grow on scales larger than the Jeans scale as $\delta \propto t^{2/3} \propto R$. Perturbations on smaller scales oscillate as acoustic waves.
- Before recombination, baryons are tightly coupled to radiation. The baryon Jeans scale is of order of the horizon size. After recombination, it drops by a factor 10^{10}.
- Silk damping reduces power on scales smaller than 40 Mpc.
- Free-streaming of HDM suppresses exponentially power on scales smaller than few Mpc (for $\Omega_\nu = 1$).
- CDM with baryons is affected only by Silk damping.

12.2.4 Results

- The three models without cosmological constant shown in Fig. 12.2 all fail.
- the exponential suppression on small scales typical for HDM is not observed, can be used to derive limit on $\Omega_\nu \lesssim 0.05$ or $\sum m_{\nu_i} \lesssim 2.2$ eV.
- acoustic baryon oscillations are only a tiny sub-dominant effect, but are now observed, cf. Fig. 12.3.

12.3 Cosmic microwave background

12.3.1 Blackbody radiation in an expanding universe

We discussed earlier that the temperatur of photons decreases as $T \sim 1/R$ in an expanding universe. Now we want to justify that the expansion preserves the thermal spectrum in the absence of particle interactions. If reactions like $\bar{f}f \to 2\gamma$ are absent or negeligible, the total number N of photons is conserved and the number density $n = N/V$ behaves as $n \propto 1/R^3$.

From the Kirchhoff-Planck distribution

$$B_\nu d\nu = \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{\frac{h\nu}{kT}} - 1} d\nu \tag{12.67}$$

describing the energy of photons emitted per time and area by a body in thermal equilibrium, we obtain the number density n of photons as $n(\nu, T)d\nu = 4\pi/(ch\nu)B_\nu(\nu, T)d\nu$.

Assume that at certain time with scale factor R the distribution of photons is thermal. Then

$$dN = V \,\frac{4}{ch\nu}\, B_\nu d\nu = \frac{8V}{c^3} \frac{h\nu^2\, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1} \tag{12.68}$$

At a different time with scale factor R', the frequency of a single photon scales as $\nu' \propto 1/R'$ and thus also the frequency interval $d\nu$ as $d\nu' \propto 1/R'$. Thus the expression for dN becomes

$$dN' = \frac{8V(R'/R)^3}{c^3} \frac{h\nu^2(R/R')^2\, d\nu(R/R')}{\exp\left(\frac{h\nu(R/R')}{kT'}\right) - 1} = \frac{8V}{c^3} \frac{h\nu^2\, d\nu}{\exp\left(\frac{h\nu(R/R')}{kT'}\right) - 1} \tag{12.69}$$

The last expression agrees with Eq. (12.68) as required by photon number conservation, $dN = dN'$, if

$$\frac{1}{T} = \frac{R}{R'}\frac{1}{T'} \qquad \text{or} \quad T = \frac{R'}{R}\, T' \tag{12.70}$$

Thus a distribution of photons remains thermal, but changes its temperature as $T \propto 1/R$. The same holds for massless fermions, but not for massive particle: Since the momentum is redshifted, $p = p_{\mathrm{f}} R/R_{\mathrm{f}}$, the distribution of massiv species has an time-dependent chemical potential after decoupling.

12.3.2 Photon recombination and decoupling

When did the universe become transparent to photons? As long as free electrons are around, Thompson scattering is the most important scattering process. Thus the universe may became transparent, when electrons and protons combine to hydrogen atoms ("recombination").

The inding energy of hydrogen is $B = m_p + m_e - m_H = 13.6$ eV. Thus the particles are non-relativstic and their number density is

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp[-\beta(m_i - \mu_i)]\,. \tag{12.71}$$

If the reaction $p + e^- \leftrightarrow H + \gamma$ is in chemical equlibrium, then $\mu_p + \mu_e = \mu_H$. Thus we can replace μ_H first by μ_e and μ_p. The latter are then re-expressed by n_e and n_p,

$$\exp(\beta\mu_H) = \exp[\beta(\mu_p + \mu_e)] = \frac{g_H}{g_p g_e}\, n_p n_e \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp[\beta(m_p + m_e - m_H)]\,. \tag{12.72}$$

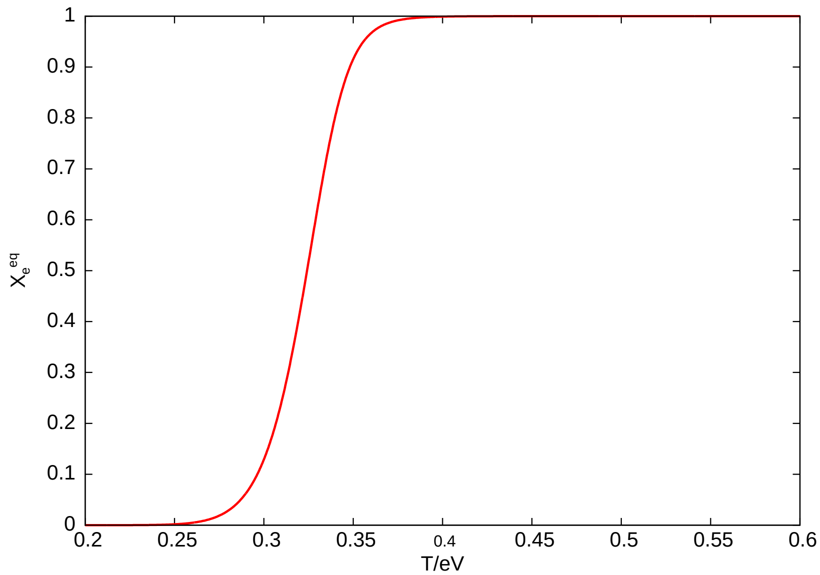


Figure 12.5: The equilibrium ionization fraction X_e^{eq} as function of T for $\eta = 6 \times 10^{-10}$.

Here set in the pre-factor $m_p \approx m_H$ and keept the exact masses only in the exponential. Inserting this expression for $\exp(\beta\mu_H)$ together with the definition of the binding energy B, $B_H = m_p + m_e - m_H$, we obtain

$$n_H = \frac{g_H}{g_p g_e}\, n_p n_e \left(\frac{2\pi}{m_e T}\right)^{3/2} \exp(B/T)\,. \tag{12.73}$$

Define the fractional equilibrium ionization X_e^{eq} as $X_e^{\text{eq}} = n_e/n_b$ with $n_b = n_p + n_H$, then charge neutrality $n_e = n_p$ implies $n_e = n_p = Xn_B$ and $n_H = (1-X)n_b$.

With $g_e = g_p = 2$, $g_H = 4$ and $n_b = \eta n_\gamma$ ($n_\gamma = \frac{2\zeta(3)}{\pi^2}\, gT^3$), we obtain the so-called "Saha equation" for the fractional equlibrium ionization

$$\frac{1 - X_e^{\text{eq}}}{(X_e^{\text{eq}})^2} = \frac{n_b n_H}{n_e n_p} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \left(\frac{T}{m_e}\right)^{3/2} \eta\, \exp(B/T) \tag{12.74}$$

The fact that $\eta \ll 1$, i.e. that the number of photons per baryon is extremly large, means that recombination (as nucelosynthesis) takes place later than naively expected.

Defining (somewhat arbitrary) recombination as the point when $X_e^{\text{eq}} = 0.1$, then one reads off $T_{\text{rec}} \approx 0.3\,\text{eV}$ from Fig. 12.5. This implies $z_{\text{rec}} \approx 1300$ from $z_{\text{rec}} = (1+z_{\text{rec}})T_{\gamma,0} \approx 3500K \approx 0.3\,\text{eV}$ and $t = 2/(3H_0)(1+z_{\text{rec}})^{-3/2}/\sqrt{\Omega_m} \approx 385.000\,\text{yr}$.

The equilibrium value is only maintained as long as the reaction $p + e^- \leftrightarrow H + \gamma$ is fast compared to the expansion rate. A calculation similar to the CDM abundance gives $X_e(\infty) \sim 3 \times 10^{-5}/(\Omega_b h)$.

Now one can calculate the optical depth of photons,

$$\tau = \exp\left(-\int dl(n_e\sigma_T + n_H\sigma_H)\right) \tag{12.75}$$

where σ_H is the cross section for photon-absorption of hydrogen and σ_T Thompson scattering of free electrons. The decoupling of photons or the position of the last scattering surface is defined by $\tau = \tau_0 = 2/3$.

12.3.3 Angular power spectrum

Fluctuations in the CMB temperature are seen on the sphere of last scattering. Thus it is convenient to decompose a map of CMB temperatures $T(\vartheta, \phi)$ into spherical harmonics $Y_{lm}(\vartheta, \phi)$,

$$T = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\vartheta, \phi) \tag{12.76}$$

The first two moments are usually excluded and considered seperately. The monopole moment $l = 0$ of a CMB temperature map corresponds to the temperature of the CMB, $T_\gamma = 2.725\,\mathrm{K}$.

The relative motion of the Sun with respect to the CMB introduces (mainly) a dipole ($l = 1$) anisotropy. More precisely,

$$T = \frac{T_0\sqrt{1-\beta^2}}{1-\beta\cos\vartheta} = T(1 + \cos\beta + \mathcal{O}(\beta^2)) \tag{12.77}$$

and the higher moments can be neglected since $\beta \ll 1$. From the size of dipole one deduced that the Sun moves with 370 km/h relative to the CMB. (The velocity of the Local Group relative to the CMB is 600 km/s.)

If this dipole anisotropy is subtracted, temperature differences are $\Delta T/T \sim 10^{-5}$ in different directions of the sky. In each direction, perfect blackbody spectrum. The moments $l \geq 2$ of the flucuations $\Delta T/T$,

$$\frac{\Delta T}{T} = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\vartheta, \phi) \tag{12.78}$$

are connected with the cosmological parameters and physics between recombination and today.

For an isotropic universe, the l dependence of the a_{lm} contains no information and one defines therefore

$$C_l = \langle a_{lm} a_{lm}^* \rangle = \frac{1}{2l+1} \sum_{m=-l}^{l} a_{lm} a_{lm}^* \,. \tag{12.79}$$

Since a single harmonic Y_{lm} corresponds roughly to angular variations of $\vartheta \sim \pi/l$, the power spectrum C_l is connected to angular scale π/l.

12.3.4 Large angular scales: Sachs-Wolfe effect

Temperature fluctuation on the sphere of last scattering, $\delta T/T|_i$, are connected to density fluctuations $\delta\rho/\rho$ and therefore to fluctations in the (Newtonian) gravitational potential $\delta\phi$. Thus the observed temperature fluctuation $\delta T/T|_f$ consist of two terms,

$$\left.\frac{\delta T}{T}\right|_f = \left.\frac{\delta T}{T}\right|_i - \Phi \,, \tag{12.80}$$

where Φ takes into account that the energy of photons that have to climb out of a potential well is gravitationally redshifted. Since an overdensity of photons results in a hot spot, $\delta T_i > 0$, but also in larger gravitational potential Φ, the two terms on the RHS of Eq. (12.80) cancel partially.

The size of the term Φ can be understood as follows: In the rest frame of the fluid fluctuation $\delta\rho$ in the density disappear. From the Poisson equation $\Delta\Phi = 4\pi G\rho$ it follows that in this frame also fluctuation in Φ are zero. Because of $T \propto 1/R$, the transformation between fluid rest-frame and CMB rest-frame corresponds to change in scale factors $\delta T/T = -\delta R/R$ or with $P = w\rho$, $R \propto t^{2/[3(1+w)]}$,

$$\left.\frac{\delta T}{T}\right|_i = -\frac{\delta R}{R} = -\frac{2}{3(1+w)}\frac{\delta t}{t}\,. \tag{12.81}$$

The time-dilation of a clock in (weak) gravitational field follows from $\mathrm{d}s = \sqrt{1-2\Phi}dt \approx (1-\Phi)\mathrm{d}t$. Hence with $\delta s = -\Phi\delta t$ and $w = 0$, we have $\delta T/T|_i = 2\phi/3$ for a matter-dominated universe. Combining both terms, we obtain

$$\left.\frac{\delta T}{T}\right|_f = -\frac{1}{3}\,\Phi\,. \tag{12.82}$$

12.3.5 Intermediate and small angular scales

Position of frist peak Let's estimate the maximal angular seperation of a region that was in causally connected at the time of recombination. Two points with anglar distance ϑ on the surface of last scattering are seperated today by the linear distance $l = d_{\mathrm{ls}}\vartheta$, where $d_{\mathrm{ls}} = c(t_0 - t_{\mathrm{ls}}) \approx ct_0$ is the distance a photon travelled freely after its last scattering at t_{ls}. Thus the maximal angular seperation of a causally connected points is with $l = ct_{\mathrm{ls}}(1+z_{\mathrm{ls}})$

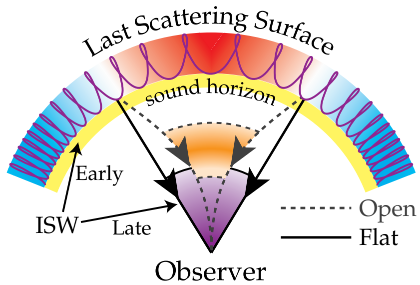
$$\vartheta = \frac{(1+z_{\mathrm{ls}})t_{\mathrm{ls}}}{t_0} \approx \frac{1100 \times 300.000\mathrm{yr}}{14 \times 10^9\mathrm{yr}} \approx 0.02 \approx 1^\circ \tag{12.83}$$

The sound horizon has approximately the same angular size $v_s \approx c/\sqrt{3}$. Its physical size serves as a ruler at the fixed redshift z_{ls} to measure the geometry of space-time. Figure 12.6 illustrates how our estimate of the angular size of the sound horizon (assuming a flat universe) would be changed by a possible curvature: In an open universe the angular scale would be smaller, in a closed universe larger than our estimate for a flat universe.

The fluid of photons and nucleons performs acoustic oscillations, for $k > k_J = \left(\frac{4\pi G\rho_0}{v_s^2}\right)^{1/2}$. On scales around and smaller than 1 degree, the Jeans frequency and its higher harmonics are visible in the power spectrum, similar to standing acoustic waves on a drum. Their relative size of peaks and precise locations gives information about cosmological parameters.

Tail at $l \gtrsim 1000$ The last scattering surface has a finite thickness d. Anisotropies that correspond to scales smaller than this thickness are damped. The physical process are adiabatic or Silk damping.

Plateau at $l \lesssim 30$ The wavelenght of perturbations with $l \lesssim 30$ was larger than the Hubble horizon at recombination. They were frozen in and could not changed by any physical process. Thus $T(k) \approx 1$ and they can be used to determine the primordial power spectrum.



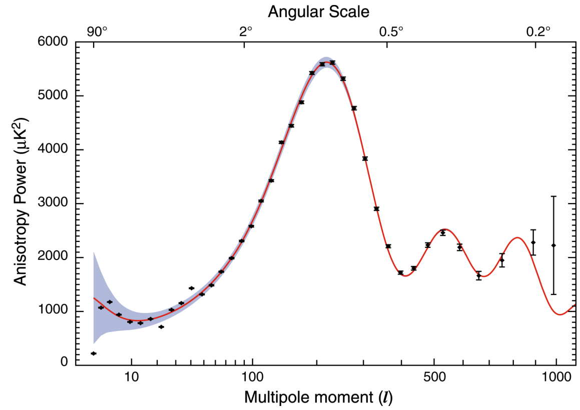


Figure 12.6: Left: The angular diameter of the sound horizon at last scattering pobes the geometry of space-time. Right: The angular power spectrum of the CMB as measued by the WMAP experiment 2006.

Summary of effect Additionally to the perturbation in T and Φ, baryons have also non-constant velocity: Density fluctuation induce by the continuity equation streaming motions in the plasam. This results in an additional contribution to the initial temperature fluctuation due to the Doppler effect,

$$\left.\frac{\delta T}{T}\right|_f = -\frac{1}{3}\,\Phi + v_{\rm radial}\,. \tag{12.84}$$

Since $v_{\rm radial}$ is maximal when $\delta\rho = 0$, the Doppler contribution fills up the zeros of $\delta T/T|_i$.

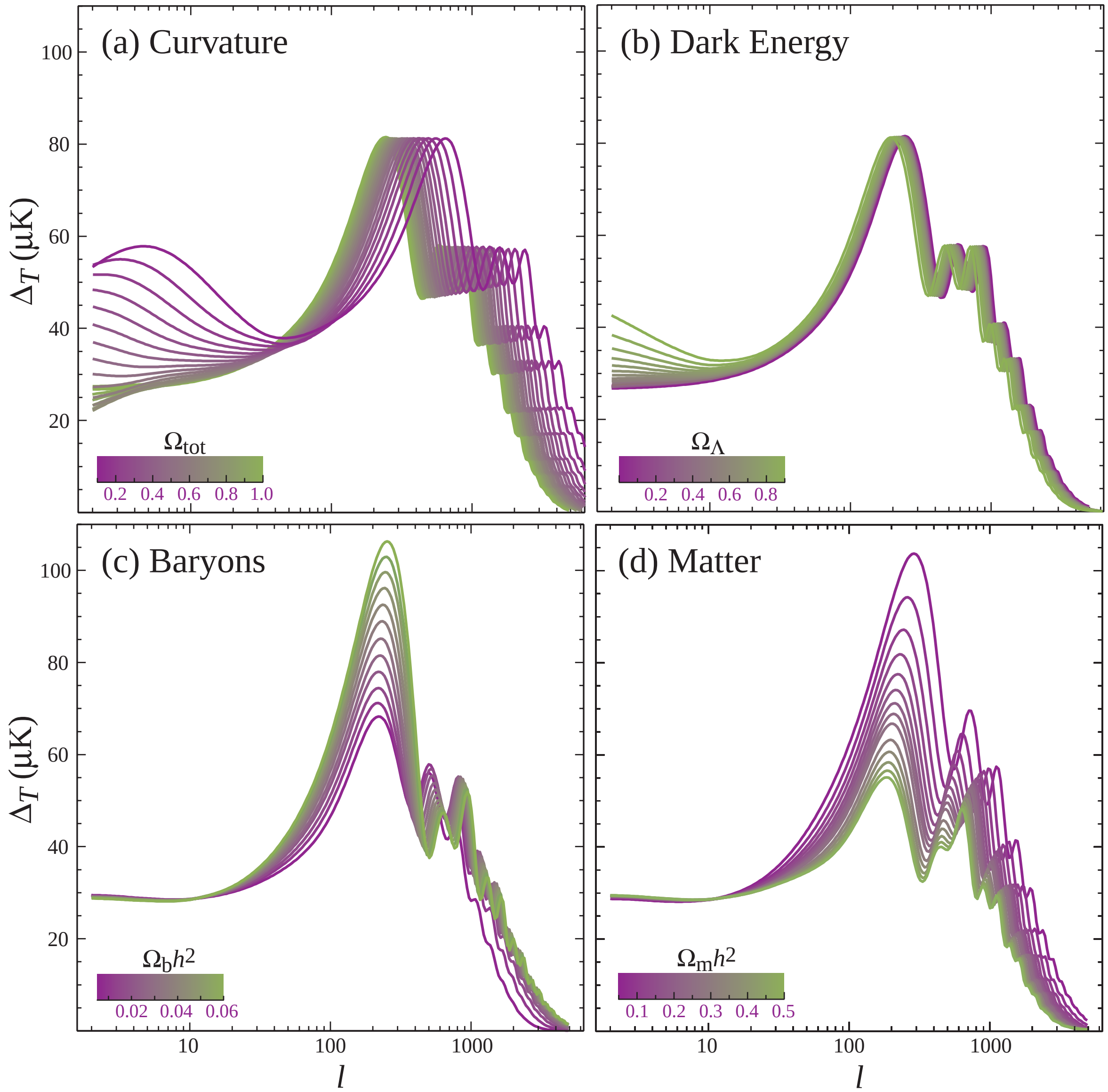


Figure 12.7: The influence of several cosmological parameters on angular power spectrum of the CMB.