Week 6: Differential geometry I

Tensor algebra

Covariant and contravariant tensors Consider two n dimensional coordinate systems x and $\tilde{x}$ and assume that we can express the x^i as functions of the $\tilde{x}^i$,

$$x^i = f(\tilde{x}^i, \ldots, \tilde{x}^n) \tag{205}$$

or more briefly $x^i = x(\tilde{x}^i)$. Forming the differentials, we obtain

$$\mathrm{d}x^i = \frac{\partial x^i}{\partial \tilde{x}^j}\,\mathrm{d}\tilde{x}^j\,. \tag{206}$$

The transformation matrix

$$a^i_j = \frac{\partial x^i}{\partial \tilde{x}^j} \tag{207}$$

is a $n \times n$ dimensional matrix with determinant ("Jacobian") $J = \det(a)$. If $J \neq 0$ in the point P, we can invert the transformation,

$$\mathrm{d}\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\,\mathrm{d}x^j = \tilde{a}^i_j\,\mathrm{d}x^j\,. \tag{208}$$

The transformation matrices are inverse to each other, $\tilde{a}^i_j a^j_k = \delta^i_k$. According to the product rule of determinants, $J(a) = 1/J(\tilde{a})$.

A *contravariant vector* (or contravariant tensor of rank one) has a n-tupel of components that transforms as

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\,X^j\,. \tag{209}$$

By definition, a scalar field ϕ remains invariant under a coordinate transformation, i.e. $\phi(x) = \phi(\tilde{x})$ at all points. Consider now the derivative of ϕ,

$$\frac{\partial \phi(x(\tilde{x}))}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i}\;\frac{\partial \phi}{\partial x^j}\,. \tag{210}$$

This is the inverse transformation matrix and we call a *covariant vector* (or covariant tensor of rank one) a n-tupel transforming as

$$\tilde{X}_i = \frac{\partial x^j}{\partial \tilde{x}^i}\;X_j\,. \tag{211}$$

More generally, we call an object T that transforms as

$$\tilde{T}^{i,\ldots,n}_{j,\ldots,m} = \underbrace{\frac{\partial \tilde{x}^i}{\partial x^{i'}} \cdots \frac{\partial \tilde{x}^n}{\partial x^{n'}}}_{n}\;\underbrace{\frac{\partial x^{j'}}{\partial \tilde{x}^j} \cdots \frac{\partial x^{m'}}{\partial \tilde{x}^m}}_{m}\;T^{i,\ldots,n}_{j,\ldots,m} \tag{212}$$

a tensor of rank (n, m).

Dual basis We defined earlier $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. Now we define a dual basis $\mathbf{e}^i$ with metric g^{ij} via

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j \,. \tag{213}$$

We want to determine the relation of g^{ij} with g_{ij}. First we set

$$\mathbf{e}^i = A^{ij}\mathbf{e}_j \,, \tag{214}$$

multiply then with $\mathbf{e}^k$ and obtain

$$g^{ik} = \mathbf{e}^i \cdot \mathbf{e}^k = A^{ij}\mathbf{e}_j \cdot \mathbf{e}^k = A^{ik} \,. \tag{215}$$

Hence the metric g^{ij} maps covariant vectors X_i into contravariants vectors X^i, while g_{ij} provides a map into the opposite direction. In the same way, we can use $\mathbf{g}$ to raise and lower indices of any tensor.

Next we multiply $\mathbf{e}^i = g^{ij}\mathbf{e}_j$ with $\mathbf{e}_k = g_{kl}\mathbf{e}^l$,

$$\delta_k^i = \mathbf{e}^i \cdot \mathbf{e}_k = \mathbf{e}^i \cdot g_{kl}\mathbf{e}^l = g_{kl}g^{ij} \tag{216}$$

or

$$\delta_k^i = g_{kl}g^{li} \,. \tag{217}$$

Thus the components of the covariant and the contravariant metric tensors, g_{ij} and g^{ij}, are inverse matrices of each other.

Example: Spherical coordinates 1:
Calculate for spherical coordinates $x = (r, \vartheta, \phi)$ in $\mathbb{R}^3$,

$$\begin{aligned} x'_1 &= r \sin\vartheta \cos\phi \,, \\ x'_2 &= r \sin\vartheta \sin\phi \,, \\ x'_3 &= r \cos\vartheta \,, \end{aligned}$$

the components of g_{ij} and g^{ij}, and $g \equiv \det(g_{ij})$.
From $\mathbf{e}_i = \partial x'^j/\partial x^i \mathbf{e}_j$, it follows

$$\begin{aligned} \mathbf{e}_1 &= \frac{x_j}{\partial r}\,\mathbf{e}'_j = \sin\vartheta\cos\phi\,\mathbf{e}'_1 + \sin\vartheta\sin\phi\,\mathbf{e}'_2 + \cos\vartheta\mathbf{e}'_3 \,, \\ \mathbf{e}_2 &= \frac{x_j}{\partial\vartheta}\,\mathbf{e}'_j = r\cos\vartheta\cos\phi\,\mathbf{e}'_1 + r\cos\vartheta\sin\phi\,\mathbf{e}'_2 - r\sin\vartheta\mathbf{e}'_3 \,, \\ \mathbf{e}_3 &= \frac{x_j}{\partial\phi}\,\mathbf{e}'_j = -r\sin\vartheta\sin\phi\,\mathbf{e}'_1 + r\sin\vartheta\cos\phi\,\mathbf{e}'_2 \,. \end{aligned}$$

Since the $\mathbf{e}_i$ are orthogonal to each other, the matrices g_{ij} and g^{ij} are diagonal. From the definition $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ one finds $g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\vartheta)$ Inverting g_{ij} gives $g^{ij} = \mathrm{diag}(1, r^{-2}, r^{-2}\sin^{-2}\vartheta)$. The determinant is $g = \det(g_{ij}) = r^4\sin^2\vartheta$. Note that the volume integral in spherical coordinates is given by

$$\int d^3x' = \int d^3x\; J = \int d^3x\,\sqrt{g} = \int \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\phi\, r^2\sin\vartheta \,,$$

since $g_{ij} = \frac{\partial x^{k\prime}}{\partial\tilde{x}^i}\frac{\partial x^{l\prime}}{\partial\tilde{x}^j}\,g'_{kl}$ and thus $\det(g) = J^2\det(g') = J^2$ with $\det(g') = 1$.

Tensor analysis

Manifolds A manifold $\mathcal{M}$ is any set that can be continuously parametrized. The number of independent parameters needed to specify uniquely any point of $\mathcal{M}$ is its dimension, the parameters are called coordinates. Examples are e.g. rigid rotations in $\mathbb{R}^3$ (with 3 Euler angles, dim = 3) or the phase space (q^i, p_i) with dim $= 2n$ of classical mechanics.

We require the manifold to be smooth: the transitions from one set of coordinates to another one, $x^i = f(\tilde{x}^i, \ldots, \tilde{x}^n)$, should be C^∞. In general, it is impossible to cover all $\mathcal{M}$ with one coordinate system that is well-defined on all $\mathcal{M}$. (Example are spherical coordinate on a sphere S^2, where ϕ is ill-defined at the poles.) Instead one has to use patches of different coordinates that (at least partially) overlap.

Doing analysis on a manifold requires an additional structure that makes it possible to compare e.g. tangent vectors living in tangent spaces at different points of the manifold: A prescription is required how a vector should be transported from point P to Q in order to calculate a derivative. Mathematically, many different schemes are possible (and sensible), but we should require the following:

- A derivative of a tensor should be again a tensor. This may require a modification of the usual partial derivative; this modification should however vanish for a flat space.
- The length of a vector should remain constant being transported along the manifold. (Think about the four velocity $|u| = 1$ or $|p| = m$.)
- A vector should not be twisted "unnecessarily" being transported along the manifold.

A Riemannian manifold is a differentiable manifold with a symmetric, positive-definite tensor-field g_{ij}. Space-time in general relativity is a four-dimensional pseudo-Riemannian (also called Lorentzian) manifold, where the metric has the signature (1,3).

Affine connection and covariant derivative

Consider how the partial derivative of a vector field, $\partial_c X^a$, transforms under a change of coordinates,

$$\partial'_c X'^a = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b} X^b\right) = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b} X^b\right) \tag{218}$$

$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\partial_d X^b + \underbrace{\frac{\partial^2 x'^a}{\partial x^b \partial x^d}\,\frac{\partial x^d}{\partial x'^c}}_{\equiv -\Gamma^a_{bc}} X^b\,. \tag{219}$$

The first term transforms as desired as a tensor of rank (1,1), while the second term—caused by the in general non-linear change of the coordinate basis—destroys the tensorial behavior. If we define a covariant derivative $\nabla_c X^a$ of a vector X^a by requiring that the result is a tensor, we should set

$$\boxed{\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b\,.} \tag{220}$$

The n^3 quantities Γ^a_{bc} ("affine connection") transform as

$$\Gamma'^a_{\ bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d_{ef} + \frac{\partial^2 x^d}{\partial x'^b \partial x'^c}\frac{\partial x'^a}{\partial x^d}\,. \tag{221}$$

For a general tensor, the covariant derivative is defined by the same reasoning as

$$\nabla_c T^{a\ldots}_{b\ldots} = \partial_c T^{a\ldots}_{b\ldots} + \Gamma^a_{dc} T^{d\ldots}_{b\ldots} + \ldots - \Gamma^d_{bc} T^{a\ldots}_{d\ldots} - \ldots \tag{222}$$

Note that it is the last index of the connection coefficients that is the same as the index of the covariant derivative. The *p*lus sign goes together with upper (su*p*erscripts), the minus with lower indices.

From the transformation law (221) it is clear that the inhomogeneous term disappears for an antisymmetric combination of the connection coefficients Γ in the lower indices. Thus this combination forms a tensor, called torsion,

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb}\,. \tag{223}$$

We consider only symmetric connections, $\Gamma^a_{bc} = \Gamma^a_{cb}$, or torsionless manifolds.

Metric connection

We assume for the moment that the partial derivative of the basis vectors can be expressed as a linear combination of the basis vectors,

$$\mathbf{e}_{k,l} = \Gamma^j_{kl}\mathbf{e}_j\,. \tag{224}$$

We differentiate the definition of the metric tensor, $g_{ab} = \mathbf{e}_a \cdot e_b$, with respect to x^c,

$$\begin{aligned}
\partial_c g_{ab} &= (\partial_c \mathbf{e}_a)\cdot\mathbf{e}_b + \mathbf{e}_a\cdot(\partial_c\mathbf{e}_b) = \Gamma^d_{ac}\mathbf{e}_d\cdot\mathbf{e}_b + \mathbf{e}_a\Gamma^d_{bc}\mathbf{e}_d = & (225)\\
&= \underline{\Gamma^d_{ac}g_{db}} + \Gamma^d_{bc}g_{ad}\,. & (226)
\end{aligned}$$

We obtain two equivalent expression by a cyclic permutation of the indices a, b, c,

$$\begin{aligned}
\partial_b g_{ca} &= \Gamma^d_{cb}g_{da} + \underbrace{\Gamma^d_{ab}g_{cd}} & (227)\\
\partial_a g_{bc} &= \underbrace{\Gamma^d_{ba}g_{dc}} + \underline{\Gamma^d_{ca}g_{bd}}\,. & (228)
\end{aligned}$$

We add the first two terms and subtract the last one. Using additionally the symmetries $\Gamma^a_{bc} = \Gamma^a_{cb}$ and $g_{ab} = g_{ba}$, the underlined terms cancel, and dividing by two we obtain

$$\frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}) = \Gamma^d_{\ cb}g_{ad}\,. \tag{229}$$

Multiplying by g^{ea} and relabeling indices gives as final result

$$\boxed{\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})\,.} \tag{230}$$

If we check the transformation law of the RHS, we find that it coincides with the one defined in Eq. (221). This equivalence relies on the assumed $\Gamma^a_{bc} = \Gamma^a_{cb}$. In general, on the RHS of Eq. (230) the torsion tensor T^a_{bc} would appear and this equation defines the *metric connection* or Christoffel symbols $\{^{bc}_{a}\}$.

We define[4]

$$\Gamma_{abc} = g_{ad}\Gamma^d_{\ bc}\,. \tag{231}$$

Thus Γ_{abc} is symmetric in the last two indices. Then it follows

$$\Gamma_{abc} = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc})\,. \tag{232}$$

Adding $2\Gamma_{abc}$ and $2\Gamma_{bac}$ gives

$$2(\Gamma_{abc} + \Gamma_{bac}) = \partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc} \tag{233}$$

$$+ \quad \partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ac} = 2\partial_c g_{ab} \tag{234}$$

or

$$\partial_c g_{ab} = \Gamma_{abc} + \Gamma_{bac}\,. \tag{235}$$

Applying the general rule for covariant derivatives, Eq. (222), to the metric,

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad} = \partial_c g_{ab} - \Gamma_{acb} - \Gamma_{bca} \tag{236}$$

and inserting Eq. (235) shows that

$$\boxed{\nabla_c g_{ab} = \nabla_c g^{ab} = 0\,.} \tag{237}$$

Hence ∇_a commutes with contracting indices,

$$\nabla_c(X^a X_a) = \nabla_c(g_{ab} X^a X^b) = g_{ab}\nabla_c(X^a X^b) \tag{238}$$

and "conserves" the norm of vectors.

Since we can choose for a flat space an Cartesian coordinate system, the connection coefficients are zero and thus $\nabla_a = \partial_a$. This suggests as general rule that physical laws valid in Minkowski space hold in general relativity, if one replace ordinary derivatives by covariant ones and η_{ij} by g_{ij}.

We derive finally a formula for the contracted connection coefficients that will be later useful. We start from the definition,

$$\Gamma^a_{ab} = \frac{1}{2} g^{ad}(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab}) = \frac{1}{2} g^{ad}\partial_b g_{ad}\,., \tag{239}$$

where we exchanged the indices a and d in the last term.

Since $\partial_c g = g g^{ab}\partial_c g_{ab}$ (cf. Lemma below), we have

$$\partial_c g = g g^{ab}(\Gamma_{abc} + \Gamma_{bac}) = g(\Gamma^b_{\ bc} + \Gamma^a_{\ ac}) = 2g\Gamma^a_{\ ac}\,. \tag{240}$$

[4] We showed that $\mathbf{g}$ can be used to raise o lower tensor indices, but Γ is not a tensor.

Thus the contraction of the connection coefficients is given by

$$\Gamma^a_{ab} = \frac{1}{2} g^{-1} \partial_b g = \frac{1}{2} \partial_b \ln |g| = \frac{1}{\sqrt{|g|}} \partial_b \sqrt{|g|} \tag{241}$$

These expressions are useful to compute the contracted covariant derivative in expressions like

$$\nabla_a T^{ab} = \partial_a X^{ab} + \Gamma^a_{ca} T^{cb} = \partial_a T^{ab} + \frac{1}{\sqrt{|g|}} (\partial_a \sqrt{|g|}) T^{ab} = \frac{1}{\sqrt{|g|}} \partial_a (\sqrt{|g|} T^{ab}) \,. \tag{242}$$

Lemma: Laplace's expansion of the determinant of an nn square matrix A expresses the determinant a as a sum of n determinants of $(n-1)(n-1)$ sub-matrices B of A (the "cofactors"),

$$a = \det(A) = \sum_{j=1}^{n} a_{ij} B_{ij}$$

Differentiating with respect to a_{ij}, we get

$$\frac{\partial a}{\partial a_{ij}} = B_{ij} \,,$$

since a_{ij} does not occur in any of the cofactors B_{ij}. If the a_{ij} depend on x, differentiating with respect to x^k gives

$$\frac{\partial a}{\partial x^k} = \frac{\partial a}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial x^k} = B_{ij} \frac{\partial a_{ij}}{\partial x^k} = a A^{-1}_{ij} \frac{\partial a_{ij}}{\partial x^k} \,.$$

(The inverse A^{-1} of the matrix A is given by $A^{-1} = B/a$.)
Applying this to the metric tensor, we obtain

$$\partial_c g = g g^{ab} \partial_c g_{ab} \,.$$

Riemannian normal coordinates

In a (pseudo-) Riemannian manifold, one can find in each point P a coordinate system, called (Riemannian) normal or geodesic coordinates, with the following properties,

$$g'_{ab}(P) = \eta_{ab} \tag{243}$$

$$\partial_{c'} g'_{ab}(P) = 0 \tag{244}$$

$$\Gamma^a_{bc}(P) = 0 \tag{245}$$

We proof it by construction. We choose new coordinates x'^a centered at P,

$$x'^a = x^a - x^a_P + \frac{1}{2} \Gamma^a_{bc} (x^b - x^b_P)(x^c - x^c_P) \tag{246}$$

Here Γ^a_{bc} are the connection coefficients in P calculated in the original coordinates x^a. We differentiate

$$\frac{\partial x'^a}{\partial x^d} = \delta^a_d + \Gamma^a_{db}(x^b - x^b_P)) \tag{247}$$

Hence $\partial x'^a/\partial x^d = \delta^a_d$ at the point P. Differentiating again,

$$\frac{\partial^2 x'^a}{\partial x^d \partial x^e} = \Gamma^a_{db}\delta^b_e = \Gamma^a_{de} \tag{248}$$

Inserting these results into the transformation law (221) of the connection coefficients, where we swap in the second term derivatives of x and x',

$$\Gamma'^a{}_{bc} = \frac{\partial x'^a}{\partial x^d}\,\frac{\partial x^f}{\partial x'^b}\,\frac{\partial x^g}{\partial x'^c}\,\Gamma^d{}_{fg} - \frac{\partial^2 x'^a}{\partial x^d \partial x^f}\,\frac{\partial x^d}{\partial x'^b}\,\frac{\partial x^f}{\partial x'^c}\,. \tag{249}$$

gives

$$\Gamma'^a{}_{bc} = \delta^a_d\,\delta^f_g\,\delta^g_c\,\Gamma^d{}_{fg} - \Gamma^a{}_{df}\,\delta^d_b\,\delta^f_c = \Gamma^a{}_{bc} - \Gamma^a{}_{bc} \tag{250}$$

or

$$\Gamma'^a_{de}(P) = 0\,. \tag{251}$$

Thus we have found a coordinate system with vanishing connection coefficients at P. By a linear transformation (that does not affect ∂g_{ab}) we can bring finally g_{ab} into the form η_{ab}.

Geodesics and parallel transport

A geodesic curve is the shortest or longest curve between two points on a manifold. Such a curve extremizes the action $S(L)$ of a free particle, $L = g_{ab}\dot{x}^a\dot{x}^b$, (setting $m = 2$ and $\dot{x} = \mathrm{d}x/d\sigma$), along the path $x^a(\sigma)$. The parameter σ plays the role of time t in the non-relativistic case, while t become part of the coordinates. The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial(\dot{x}^c)} - \frac{\partial L}{\partial x^c} = 0 \tag{252}$$

Only g depends on x and thus $\partial L/\partial x^c = g_{ab,c}\dot{x}^a\dot{x}^b$. With $\partial\dot{x}^a/\partial\dot{x}^b = \delta^a_b$ we obtain

$$g_{ab,c}\dot{x}^a\dot{x}^b = 2\,\frac{\mathrm{d}}{\mathrm{d}\sigma}\,(g_{ac}\dot{x}^a) = 2(g_{ac,b}\dot{x}^a\dot{x}^b + g_{ac}\ddot{x}^a) \tag{253}$$

or

$$g_{ac}\ddot{x}^a + \frac{1}{2}(2g_{ac,b} - g_{ab,c})\dot{x}^a\dot{x}^b = 0 \tag{254}$$

Next we rewrite the second term as

$$2g_{ca,b}\dot{x}^a\dot{x}^b = (g_{ca,b} + g_{cb,a})\dot{x}^a\dot{x}^b \tag{255}$$

multiply everything by g^{dc} and obtain

$$\ddot{x}^d + \frac{1}{2} g^{dc}(g_{ab,c} + g_{ac,b} - g_{ab,c})\dot{x}^a \dot{x}^b = 0\,. \tag{256}$$

We recognize the definition of the connection coefficients and rewrite the equation of a geodesics as

$$\boxed{\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a \dot{x}^b = 0\,.} \tag{257}$$

The connection coefficients are automatically symmetric, since Minkowski space (or $\mathbb{R}^n$) is torsionfree.

In an inertial system, an alternative definition of a geodesics as the "straightest line" is the path generated by propagating its tangent vector parallel to itself. A natural generalization of this it that the covariant derivative of a unit tangent vector $u^a = \mathrm{d}x^a/d\sigma$ along $\mathbf{u}$ is zero,

$$(\nabla_{\mathbf{u}}\mathbf{u})^a = u^b \left(\frac{\partial u^a}{\partial x^b} + \Gamma^a_{bc} u^c \right) = 0\,. \tag{258}$$

We form analogous to the normal differential $\mathrm{d}f = \partial_i f \, \mathrm{d}x^i$ of a function $f(x)$ the absolute differential

$$\mathrm{D}f = \nabla_i f \mathrm{d}x^i \tag{259}$$

Vector differential operators

The gradient of a scalar function ϕ has as components

$$\nabla^i \phi = \partial^i \phi = g^{ij}\partial_j \phi = g^{ij}\frac{\partial \phi}{\partial x^j}\,. \tag{260}$$

The covariant divergence of a vector field with components X^i is

$$\nabla_i X^i = \partial_i X^i + \Gamma^i_{ki} X^k = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}X^a)\,. \tag{261}$$

The Laplace operator $\Delta \equiv \nabla_a \nabla^a$ of a scalar is

$$\Delta\phi = \nabla_a \nabla^a \phi = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}g^{ab}\partial_b \phi)\,. \tag{262}$$

Example: Spherical coordinates 3:
Calculate for spherical coordinates $x = (r, \vartheta, \phi)$ in $\mathbb{R}^3$ the gradient, divergence, and the Laplace operator. Note that one uses normally normalized unit vectors in case of a diagonal metric: this corresponds to a rescaling of vector components $V^i \to V^i/\sqrt{g_{ii}}$ (no summation in i) or basis vectors.
We express the gradient of a scalar function f first as

$$\partial^i f \, \mathbf{e}_i = g^{ij}\frac{\partial f}{\partial x^j}\,\mathbf{e}_i = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \phi}\,\mathbf{e}_\phi + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial f}{\partial \vartheta}\,\mathbf{e}_\vartheta$$

and rescale then the basis, $\mathbf{e}_i^* = \mathbf{e}_i/\sqrt{g_{ii}}$, or $\mathbf{e}_r^* = \mathbf{e}_r$, $\mathbf{e}_\phi^* = r\mathbf{e}_\phi$, and $\mathbf{e}_\vartheta^* = r\sin\vartheta\mathbf{e}_\vartheta$. In this new ("physical") basis, the gradient is given by

$$\partial^i f\,\mathbf{e}_i^* = \frac{\partial f}{\partial r}\,\mathbf{e}_r^* + \frac{1}{r}\,\frac{\partial f}{\partial\vartheta}\,\mathbf{e}_\vartheta^* + \frac{1}{r\sin\vartheta}\,\frac{\partial f}{\partial\phi}\,\mathbf{e}_\phi^*\,.$$

The covariant divergence of a vector field with rescaled components $X^i/\sqrt{g_{ii}}$ is with $\sqrt{g} = r^2\sin\vartheta$ given by

$$\begin{aligned}
\nabla_i X^i &= \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i) = \frac{1}{r^2\sin\vartheta}\left(\frac{\partial(r^2\sin\vartheta X_r)}{\partial r} + \frac{\partial(r^2\sin\vartheta X_\vartheta)}{r\partial\vartheta} + \frac{\partial(r^2\sin\vartheta X_\phi)}{r\sin\vartheta\partial\phi}\right)\\
&= \frac{1}{r^2}\,\frac{\partial(r^2 X_r)}{\partial r} + \frac{1}{r\sin\vartheta}\,\frac{\partial(\sin\vartheta X_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\,\frac{\partial X_\phi}{\partial\phi}\\
&= \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)X_r + \left(\frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{r}\right)X_\phi + \frac{1}{r\sin\vartheta}\,\frac{\partial X_\phi}{\partial\phi}\,.
\end{aligned}$$