THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF PHYSICS

Contact person:
Name: Jan Myrheim
Telephone: 73 59 36 53 (mobil 90 07 51 72)

**Examination, course FY8104 Symmetry in physics**
Wednesday December 9, 2009
Time: 09.00–13.00

Grades made public: Wednesday December 30, 2009

Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

## Problem 1:

A group $G$ of order 12 has the following multiplication table.

| | $e$ | $a$ | $b$ | $c$ | $s$ | $t$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $s$ | $t$ | $u$ | $v$ | $w$ | $x$ | $y$ | $z$ |
| $a$ | $a$ | $e$ | $c$ | $b$ | $t$ | $s$ | $v$ | $u$ | $x$ | $w$ | $z$ | $y$ |
| $b$ | $b$ | $c$ | $e$ | $a$ | $u$ | $v$ | $s$ | $t$ | $y$ | $z$ | $w$ | $x$ |
| $c$ | $c$ | $b$ | $a$ | $e$ | $v$ | $u$ | $t$ | $s$ | $z$ | $y$ | $x$ | $w$ |
| $s$ | $s$ | $u$ | $v$ | $t$ | $w$ | $y$ | $z$ | $x$ | $e$ | $b$ | $c$ | $a$ |
| $t$ | $t$ | $v$ | $u$ | $s$ | $x$ | $z$ | $y$ | $w$ | $a$ | $c$ | $b$ | $e$ |
| $u$ | $u$ | $s$ | $t$ | $v$ | $y$ | $w$ | $x$ | $z$ | $b$ | $e$ | $a$ | $c$ |
| $v$ | $v$ | $t$ | $s$ | $u$ | $z$ | $x$ | $w$ | $y$ | $c$ | $a$ | $e$ | $b$ |
| $w$ | $w$ | $z$ | $x$ | $y$ | $e$ | $c$ | $a$ | $b$ | $s$ | $v$ | $t$ | $u$ |
| $x$ | $x$ | $y$ | $w$ | $z$ | $a$ | $b$ | $e$ | $c$ | $t$ | $u$ | $s$ | $v$ |
| $y$ | $y$ | $x$ | $z$ | $w$ | $b$ | $a$ | $c$ | $e$ | $u$ | $t$ | $v$ | $s$ |
| $z$ | $z$ | $w$ | $y$ | $x$ | $c$ | $e$ | $b$ | $a$ | $v$ | $s$ | $u$ | $t$ |

It is generated for example by the three elements $a, b, s$ with the relations $a^2 = b^2 = s^3 = e$, $ab = ba$, $sa = bs$, $sb = cs$, $sc = as$.

**a)** Find subgroups of $G$.

**b)** Find the conjugation classes.

**c)** If you find a normal (invariant) subgroup $H$, find also the multiplication table of the quotient group $G/H$.

**d)** Find the character table.

Hint: A representation of a quotient group $G/H$ is also representation of $G$.

The following orthogonality relations hold for a finite group of order $N$.

Let $\chi_i^{(\mu)}$ be the character value of the conjugation class $i$, with $N_i$ elements, in the irreducible representation $\mu$. Then

$$\sum_i N_i (\chi_i^{(\mu)})^* \chi_i^{(\nu)} = N\,\delta_{\mu\nu}\ ,$$

$$\sum_\mu (\chi_i^{(\mu)})^* \chi_j^{(\mu)} = \frac{N}{N_i}\,\delta_{ij}\ .$$

**e)** The group $G$ is the same as $A_4$, the alternating group of degree 4, which is the subgroup of even permutations in the symmetric group $S_4$. To see this, take for example $a = (12)(34)$, $b = (14)(23)$, $s = (123)(4)$.

$A_4$ is also the subgroup of proper rotations, excluding reflections, in the symmetry group of a regular tetrahedron, for example a methane ($CH_4$) molecule.

The orbital angular momentum quantum number $\ell = 0, 1, 2, \ldots$ labels the irreducible representations of the full rotation group SO(3). The dimension of an irreducible representation is $2\ell + 1$, and the character value as a function of the rotation angle $\alpha$ is

$$\chi^{(\ell)}(\alpha) = \frac{\sin((\ell + \frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})}\ .$$

An irreducible representation of SO(3) is in general a reducible representation of the subgroup $A_4 \subset \mathrm{SO}(3)$.

How does the irreducible representation of SO(3) with $\ell = 2$ split into irreducible representations of $A_4$?

Some formulae that may be useful in the following:

$$[A, BC] = [A, B]\, C + B\, [A, C] \qquad \text{(the Leibniz rule for commutation)}$$

$$\mathrm{e}^{A} B \mathrm{e}^{-A} = B + [A, B] + \frac{1}{2}\, [A, [A, B]] + \cdots + \frac{1}{n!}\, [A, [A, \cdots [A, B] \cdots]] + \cdots$$

## Problem 2:

The Hamiltonian of a one dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}\, m\omega^2 x^2 \ ,$$

where $m$ is the mass and $\omega$ the angular frequency. For simplicity we set $m = 1$, $\omega = 1$, and $\hbar = 1$ (this is a question of choosing a convenient set of units). Then

$$H = \frac{1}{2}\,(p^2 + x^2) = a^\dagger a + \frac{1}{2} \ ,$$

where

$$a = \frac{1}{\sqrt{2}}\,(x + \mathrm{i}p) \ , \qquad a^\dagger = \frac{1}{\sqrt{2}}\,(x - \mathrm{i}p) \ . \tag{1}$$

The position $x$ and momentum $p$ are Hermitean linear operators satisfying the canonical commutation relation $[x, p] = \mathrm{i}$, or equivalently, $[a, a^\dagger] = 1$.

Let $|\psi\rangle$ be a state vector, normalized so that $\langle\psi|\psi\rangle = 1$. The expectation value of $H$ in the state $|\psi\rangle$ is

$$\langle H\rangle = \langle\psi|H|\psi\rangle = \langle\phi|\phi\rangle + \frac{1}{2} \ ,$$

where $|\phi\rangle = a\,|\psi\rangle$. The ground state $|0\rangle$, in which $\langle H\rangle$ is minimal, is given by the equation

$$a\,|0\rangle = 0 \ .$$

In the quantization of the electromagnetic field we describe one single mode of the field with angular frequency $\omega$ as a one dimensional harmonic oscillator. Then $a^\dagger$ and $a$ are the creation and annihilation operators of photons, and $|0\rangle$ is the vacuum state with no photons in this mode.

**a)** Define

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n\,|0\rangle \quad \text{for} \quad n = 1, 2, \ldots \ .$$

Show that $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, and that $N\,|n\rangle = n\,|n\rangle$, where $N = a^\dagger a = aa^\dagger - 1$ is the (photon) number operator.

Show also that the state $|n\rangle$ is normalized, $\langle n|n\rangle = 1$.

**b)** Let $z$ be an arbitrary complex number, and define a "displacement operator"

$$D = D(z) = \mathrm{e}^{za^\dagger - z^* a} \ .$$

Show that $D$ is unitary, $D^\dagger = D^{-1}$, and that $DaD^{-1} = a - z$.

The operator $D$ transforms the ground state $|0\rangle$ into the state $|z\rangle = D\,|0\rangle$, which is called a coherent state. Show that the coherent state $|z\rangle$ is an eigenstate with eigenvalue $z$ of the non-Hermitean operator $a$. That is, $a\,|z\rangle = z\,|z\rangle$, or equivalently,

$$(a - z)\,|z\rangle = 0 \ .$$

**c)** Show that the coherent state can be expanded in terms of the energy eigenstates as

$$|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle \ . \tag{2}$$

It is enough to show that this state solves the eigenvalue equation $a\,|z\rangle = z\,|z\rangle$, and that it is normalized.

A different method is to use the Campbell–Baker–Hausdorff formula

$$\exp(A)\exp(B) = \exp\Big(A + B + \frac{1}{2}\,[A,B] + \ldots\Big) \ ,$$

where the terms left out are commutators of commutators, in order to compute the product $\mathrm{e}^{za^\dagger}\mathrm{e}^{-z^*a}$ and thus derive an alternative formula for $D = \mathrm{e}^{za^\dagger - z^*a}$.

**d)** Assume that the state of the oscillator at time $t = 0$ is the coherent state $|z\rangle$, that $|\psi(0)\rangle = |z\rangle$. Then the state at time $t$ is $|\psi(t)\rangle = U\,|\psi(0)\rangle = U\,|z\rangle$, where $U$ is the time development operator,

$$U = U(t) = \mathrm{e}^{-\mathrm{i}tH} \ .$$

Show that $U\,|z\rangle = \mathrm{e}^{-\frac{\mathrm{i}t}{2}}\,|\mathrm{e}^{-\mathrm{i}t}z\rangle$.

Thus, a coherent state remains a coherent state as it develops in time.

Again there are at least two possible ways to solve the problem. We may either use directly equation (2), or we may compute

$$UD = UDU^{-1}U = \mathrm{e}^{zUa^\dagger U^{-1} - z^* UaU^{-1}}\,U \ .$$

**e)** The variance of $x$ in a state $|\psi\rangle$ is defined as

$$(\Delta x)^2 = \langle\psi|\,(x - \langle x\rangle)^2\,|\psi\rangle = \langle\,(x - \langle x\rangle)^2\,\rangle = \langle x^2\rangle - \langle x\rangle^2 \ .$$

The variance of $p$ is defined in a similar way. The Heisenberg uncertainty relation

$$\Delta x\,\Delta p \geq \frac{1}{2}$$

may be proved as follows. Define

$$b = x + \mathrm{i}\lambda p \ , \qquad b^\dagger = x - \mathrm{i}\lambda p \ ,$$

with $\lambda$ as a real parameter, and define also

$$w = \langle\psi|b|\psi\rangle = \langle b\rangle = \langle x\rangle + \mathrm{i}\lambda\langle p\rangle \ .$$

Since

$$\begin{aligned}(b^\dagger - w^*)(b - w) &= (x - \langle x\rangle)^2 + \lambda^2(p - \langle p\rangle)^2 + \mathrm{i}\lambda\,[\,x - \langle x\rangle\,,\,p - \langle p\rangle\,] \\ &= (x - \langle x\rangle)^2 + \lambda^2(p - \langle p\rangle)^2 - \lambda \ ,\end{aligned}$$

we may define $|\phi\rangle = (b - w)\,|\psi\rangle$ and deduce that

$$\begin{aligned}0 &\le \langle\phi|\phi\rangle = \langle\psi|(b^\dagger - w^*)(b - w)|\psi\rangle = (\Delta x)^2 + \lambda^2(\Delta p)^2 - \lambda \\ &= (\Delta x)^2 + \left(\lambda\,\Delta p - \frac{1}{2\,\Delta p}\right)^2 - \frac{1}{4\,(\Delta p)^2} \ .\end{aligned}$$

Given the state $|\psi\rangle$, the inequality must hold for an arbitrary value of $\lambda$. In particular, with $\lambda = 1/(2(\Delta p)^2)$ we obtain the Heisenberg uncertainty relation.

We say that $|\psi\rangle$ is a *minimum uncertainty state* if $\Delta x\,\Delta p = 1/2$. A minimum uncertainty quantum state is the best possible approximation to a classical state.

In the notation used above, if $|\psi\rangle$ is a minimum uncertainty state we have that

$$\langle\phi|\phi\rangle = \left(\lambda\,\Delta p - \frac{1}{2\,\Delta p}\right)^2 \ .$$

Hence, if we choose $\lambda = 1/(2(\Delta p)^2$ we must have that $|\phi\rangle = (b - w)\,|\psi\rangle = 0$.

We see that a minimum uncertainty state $|\psi\rangle = |\lambda, w\rangle$ is characterized by a real parameter $\lambda$ and a complex parameter $w$, and it is an eigenvector of the non-Hermitean operator $b = x + \mathrm{i}\lambda p$ with $w$ as eigenvalue,

$$b\,|\lambda, w\rangle = w\,|\lambda, w\rangle \ .$$

The expectation values and variances of $x$ and $p$ in this state are given by the formulae

$$\langle x\rangle + \mathrm{i}\lambda\,\langle p\rangle = w \ , \qquad \Delta x = \sqrt{\frac{\lambda}{2}} \ , \qquad \Delta p = \frac{1}{\sqrt{2\lambda}} \ .$$

A coherent state is a minimum uncertainty state with $\lambda = 1$. A minimum uncertainty state with $\lambda \neq 1$ is called a *squeezed state*, because it has smaller uncertainty for either $x$ or $p$ than a coherent state.

The squeezed state corresponding to the ground state $|0\rangle$ (the "squeezed vacuum") is the state $|\lambda, 0\rangle$ with $w = 0$, defined by the equation

$$b\,|\lambda, 0\rangle = (x + \mathrm{i}\lambda p)\,|\lambda, 0\rangle = 0 \ .$$

Compute $UbU^{-1}$ where $U = \mathrm{e}^{-\mathrm{i}tH}$ is the time development operator.

What can you say from this about the time development of the squeezed vacuum state $|\lambda, 0\rangle$?