THE NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF PHYSICS

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Examination, course FY8104 Symmetry in physics
Wednesday August 15, 2012
Time: 09.00–13.00

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Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

Problem 1:

a) Define the groups SO(4) (the four dimensional rotation group) and SO(1, 3) (the Lorentz group). That is, write down the conditions satisfied by a 4×4 matrix $\mathbf{A} \in \mathrm{SO}(4)$, and the conditions satisfied by a 4×4 matrix $\mathbf{B} \in \mathrm{SO}(1,3)$.

b) What is the relation between the Lie groups SO(4) and SO(1, 3), on the one hand, and the Lie algebras so(4) and so(1, 3), on the other hand?

Write down the conditions satisfied by a 4×4 matrix $\mathbf{X} \in \mathrm{so}(4)$, and the conditions satisfied by a 4×4 matrix $\mathbf{Y} \in \mathrm{so}(1,3)$.

Write down also the most general forms for $\mathbf{X} \in \mathrm{so}(4)$ and $\mathbf{Y} \in \mathrm{so}(1,3)$.

Problem 2:

a) According to Noether's theorem there is a relation between a conservation law in a physical system and a continuous symmetry of the system. Give a precise statement of Noether's theorem in the case of a quantum mechanical system.

b) A non-commutative (non-Abelian) continuous symmetry group, such as the three dimensional rotation group SO(3), implies that some energy levels are degenerate. Why?

Problem 3:

The symmetry group of a football is a large discrete subgroup of the three dimensional orthogonal group. Here we will consider the group of proper rotations (of determinant $+1$), excluding the reflections (of determinant -1). Call this group F (for football).

The football is sewn together from 12 regular pentagons and 20 regular hexagons. Apart from the identity transformation I we recognize three types of rotational symmetries:

a) about an axis through the centre of a pentagon;
b) about an axis through the centre of a hexagon;
c) about an axis through the midpoint of a line (edge) where two hexagons meet.

There are 60 corners (vertices) where one pentagon and two hexagons meet. Given any corner there is always exactly one proper rotation taking this corner into any other corner. Hence there are altogether 60 group elements, in the following 5 conjugation classes:

$C_1 = \{ I \}$;
$C_2 = \{$ 15 rotations of 180° (type c) $\}$;
$C_3 = \{$ 20 rotations of 120° (type b) $\}$;
$C_4 = \{$ 12 rotations of 72° (type a) $\}$;
$C_5 = \{$ 12 rotations of 144° (type a) $\}$.

a) If G is a finite group, if the group element $g \in G$ is conjugate to its inverse g^{-1}, and if χ is the character of some linear representation of G, then $\chi(g)$ is real. Explain why.

Furthermore, if the representation has dimension n, and if g is of order m, then

$$\chi(g) = \sum_{i=1}^{n} x_i \qquad \text{where} \qquad (x_i)^m = 1 \quad \text{for} \quad i = 1, 2, \ldots, n \ .$$

Explain why.

If either $m = 2$, $m = 3$, or $m = 4$, and if g is conjugate to g^{-1}, then $\chi(g)$ must be an integer. Explain why.

b) The table below gives the characters of three irreducible representations of the "football group" F. The notation $C_i(N_i)$ means that the conjugation class C_i has N_i elements.

	$C_1(1)$	$C_2(15)$	$C_3(20)$	$C_4(12)$	$C_5(12)$
$\chi^{(a)}$	1	1	1	1	1
$\chi^{(b)}$	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\chi^{(c)}$	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\chi^{(d)}$					
$\chi^{(e)}$					

Complete the character table, using the theorem from 3a) and the following orthogonality relations, which hold for a finite group of order N.

Let $\chi_i^{(\mu)}$ be the character value of the conjugation class C_i, with N_i elements, in the irreducible representation μ. Then

$$\sum_i N_i\,(\chi_i^{(\mu)})^*\,\chi_i^{(\nu)} = N\,\delta_{\mu\nu}\ ,$$

$$\sum_\mu (\chi_i^{(\mu)})^*\,\chi_j^{(\mu)} = \frac{N}{N_i}\,\delta_{ij}\ .$$

c) The orbital angular momentum quantum number $\ell = 0, 1, 2, \ldots$ labels the irreducible representations of the full rotation group SO(3).

In an irreducible representation of SO(3) the character value as a function of the rotation angle α is

$$\chi^{(\ell)}(\alpha) = \frac{\sin((\ell+\frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})}\ .$$

Calculate the dimension of the irreducible representation of SO(3) by taking the limit $\alpha \to 0$.

An irreducible representation of SO(3) is in general a reducible representation of the subgroup $F \subset \mathrm{SO}(3)$.

How does the irreducible representations of SO(3) with $\ell = 1$, $\ell = 2$, and $\ell = 3$ split into irreducible representations of F?

Some trigonometric facts:

$$\sin(36^\circ) = \sin(144^\circ) = \frac{\sqrt{10 - 2\sqrt{5}}}{4}\ , \qquad \sin(72^\circ) = \sin(108^\circ) = \frac{\sqrt{10 + 2\sqrt{5}}}{4}\ .$$

Note that it is quite possible to do the calculations numerically.

d) The C_{60} molecule, the famous "buckyball", has the carbon atoms arranged on a spherical surface like the vertices of the football.

An electron moving freely on the surface of a sphere of radius R has (kinetic) energy

$$E = E_\ell = \frac{\ell(\ell+1)\hbar^2}{2mR^2}\ .$$

Consider the four lowest energy levels, with $\ell = 0, 1, 2, 3$.

What is the degeneracy of each energy level?

How are these energy levels split by the Coulomb potential from the carbon atoms of the C_{60} molecule?

We disregard the effects of the spin of the electron.