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Eksamensoppgave i FY8104 / FY3105 Symmetrigrupper i fysikken

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NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR FYSIKK

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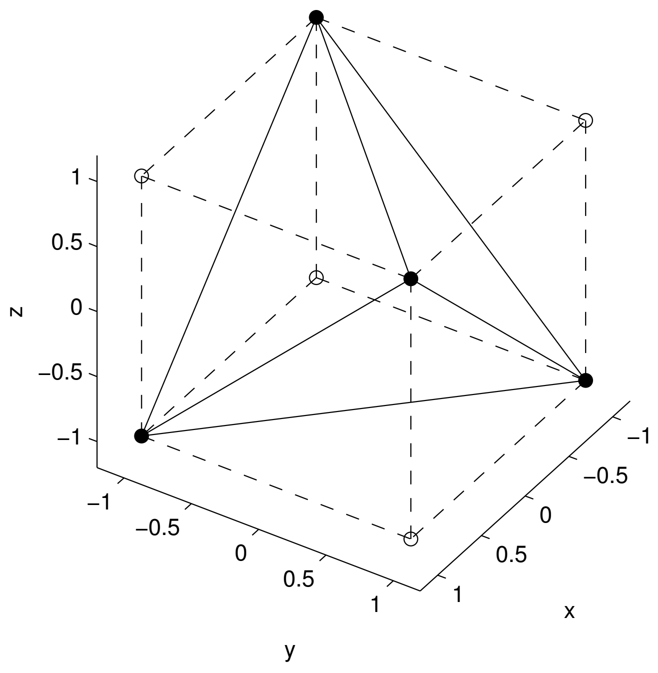
Eksamen i fag FY8104/FY3105 Symmetri i fysikken
Tirsdag 15. desember 2015
Tid: 09:00–13:00

Sensurfrist: Fredag 15. januar 2016

Tillatte hjelpemidler: Kalkulator, matematiske tabeller.

Alle deloppgaver teller likt ved sensuren.

Oppgave 1:



Figur 1: Hjørnene til et regulært tetraeder er halvparten av hjørnene til en terning.

De åtte punktene $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ i $\mathbb{R}^3$ er hjørnene til en terning. Delmengden av punkter med $xyz = 1$ er hjørnene til et regulært tetraeder. Se figur 1.

Hjørnene til tetraederet er

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \tag{1}$$

De andre hjørnene til terningen er $-\mathbf{a}, -\mathbf{b}, -\mathbf{c}, -\mathbf{d}$, de er hjørnene til et annet tetraeder.

a) Matrisen

$$\mathbf{M}_{\text{ab}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

tilhører Lie-gruppen O(3), men ikke SO(3). Forklar.

Den transformerer tetraederet som følger:

$$\mathbf{M}_{\text{ab}}\mathbf{a} = \mathbf{b}\,, \qquad \mathbf{M}_{\text{ab}}\mathbf{b} = \mathbf{a}\,, \qquad \mathbf{M}_{\text{ab}}\mathbf{c} = \mathbf{c}\,, \qquad \mathbf{M}_{\text{ab}}\mathbf{d} = \mathbf{d}\,.$$

Hva er den geometriske tolkningen, når vi snakker om rotasjoner og refleksjoner?

Notasjonen indikerer at $\mathbf{M}_{\text{ab}}$ bytter om hjørnene $\mathbf{a}$ og $\mathbf{b}$ til tetraederet. Den er en av seks transformasjonsmatriser som bytter om to hjørner uten å flytte de to andre hjørnene. Finn de andre fem matrisene.

b) Matrisene $\mathbf{M}_{\text{abc}} = \mathbf{M}_{\text{ab}}\mathbf{M}_{\text{bc}}$ og $\mathbf{M}_{\text{abcd}} = \mathbf{M}_{\text{ab}}\mathbf{M}_{\text{bc}}\mathbf{M}_{\text{cd}}$ representerer også symmetritransformasjoner til tetraederet.

Hvordan permuterer disse to transformasjonene hjørnene til tetraederet?

Hvordan tolkes de geometrisk, som rotasjoner og refleksjoner?

c) Symmetrigruppen til tetraederet består av alle mulige permutasjoner av de fire hjørnene, derfor er den isomorf med den symmetriske gruppen S_4.

Vi har her funnet en tredimensjonal representasjon av S_4. Er den irredusibel?

Forklar hvordan svaret følger av karaktertabellen til S_4, gitt her.

	1	3	8	6	6
	1^4	2^2	31	21^2	4
χ_1	1	1	1	1	1
χ_2	3	−1	0	1	−1
χ_3	2	2	−1	0	0
χ_4	3	−1	0	−1	1
χ_5	1	1	1	−1	−1

Forklar notasjonen som brukes i tabellen for konjugasjonsklassene til S_4.

Antallet elementer i hver konjugasjonsklasse er gitt i tabellen.

d) Se nå på symmetrigruppen til terningen.

Hva er ordenen til (antallet elementer i) denne gruppen?

Hvordan kan du utvide symmetrigruppen til tetraederet slik at du får symmetrigruppen til terningen? Svaret er kanskje enklere enn du tror.

Skriv opp karaktertabellen til symmetrigruppen til terningen.

Oppgave 2:

Metan, CH_4, er en virkningsfull drivhusgass fordi den absorberer infrarød stråling med frekvenser som eksiterer vibrasjoner av molekylet.

Et molekyl av metan i grunntilstanden har form av et regulært tetraeder.
Anta at karbonatomet er plasert i origo og de fire hydrogenatomene i posisjonene $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ som definert i ligning (1) og i figur 1.

La, for eksempel, $\mathbf{R}$ være symmetritransformasjonen til tetraederet slik at $\mathbf{Ra} = \mathbf{b}$, $\mathbf{Rb} = \mathbf{c}$, $\mathbf{Rc} = \mathbf{a}$, $\mathbf{Rd} = \mathbf{d}$. Siden hydrogenatomene er identiske, er denne transformasjonen en fysisk identitetstransformasjon av molekylet i grunntilstanden.

Se på en liten deformasjon av molekylet der hydrogenatomet i $\mathbf{x}$ beveger seg til $\mathbf{x} + \Delta\mathbf{x}$, med $\mathbf{x} = \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Da er $\mathbf{R}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{Rx} + \mathbf{R}\Delta\mathbf{x}$. Siden $\mathbf{Rc} = \mathbf{a}$, for eksempel, og hydrogenatomene er identiske, er dette den samme fysiske transformasjonen som

$$\begin{pmatrix} \Delta\mathbf{a} \\ \Delta\mathbf{b} \\ \Delta\mathbf{c} \\ \Delta\mathbf{d} \end{pmatrix} \stackrel{\mathbf{R}}{\longmapsto} \begin{pmatrix} \mathbf{R}\Delta\mathbf{c} \\ \mathbf{R}\Delta\mathbf{a} \\ \mathbf{R}\Delta\mathbf{b} \\ \mathbf{R}\Delta\mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a} \\ \Delta\mathbf{b} \\ \Delta\mathbf{c} \\ \Delta\mathbf{d} \end{pmatrix} .$$

Det vil si at vi representerer 3×3–matrisen $\mathbf{R}$ med 12×12–matrisen

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbf{R} .$$

For å summere opp: en permutasjon $p \in S_4$ representeres som en 4×4–matrise $\mathbf{D}_\alpha(p)$, en 3×3–matrise $\mathbf{D}_\beta(p)$, og en 12×12–matrise $\mathbf{D}(p) = \mathbf{D}_\alpha(p) \otimes \mathbf{D}_\beta(p)$. De tre transposisjonene $T_1 = (12)$, $T_2 = (23)$ og $T_3 = (34)$ genererer S_4 og representeres som følger:

$$\mathbf{D}_\alpha(T_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\alpha(T_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\alpha(T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}_\beta(T_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{D}_\beta(T_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\beta(T_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

a) Hvordan dekomponeres representasjonene $\mathbf{D}_\alpha$, $\mathbf{D}_\beta$ og $\mathbf{D} = \mathbf{D}_\alpha \otimes \mathbf{D}_\beta$ av S_4 i irredusible representasjoner?

Karaktertabellen til S_4 er gitt under deloppgave 1c).

b) Metanmolekylet har tre rotasjonsmoder og ni vibrasjonsmoder, der hydrogenatomene beveger seg, mens massesentret er i ro. Det har også tre translasjonsmoder, som ikke interesserer oss her, der karbonatomet og de fire hydrogenatomene beveger seg sammen.

Hvor mange *forskjellige* vibrasjonsfrekvenser har det?
Begrunn svaret kort.

Oppgave 3:

I denne oppgaven bruker vi «naturlige enheter» der $c = 1$ og $\hbar = 1$. Matrisene

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$\boldsymbol{\kappa}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \boldsymbol{\kappa}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \boldsymbol{\kappa}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

er generatorer for Lorentz-transformasjoner, inkludert rotasjoner.
De oppfyller kommutasjonsrelasjonene

$$[\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j] = \epsilon_{ijk}\boldsymbol{\lambda}_k , \qquad [\boldsymbol{\lambda}_i, \boldsymbol{\kappa}_j] = \epsilon_{ijk}\boldsymbol{\kappa}_k , \qquad [\boldsymbol{\kappa}_i, \boldsymbol{\kappa}_j] = -\epsilon_{ijk}\boldsymbol{\lambda}_k .$$

Kvantetilstander til en partikkel med masse $m > 0$ og spinn s kan genereres ved Lorentz-transformasjoner fra tilstander der partikkelen er i ro.

En partikkel med masse $m = 0$ har ikke noe hvilesystem, og vi må velge andre tilstander enn hviletilstandene som referanse, for eksempel tilstander der partikkelen beveger seg i positiv z-retning med firer-impuls

$$k^\mu = (k^0, k^1, k^2, k^3) = (k, 0, 0, k) , \qquad k > 0 . \tag{2}$$

Mer generelle tilstander lager vi ved å Lorentz-transformere disse referansetilstandene.

a) Definer $\boldsymbol{\mu}_1 = \boldsymbol{\lambda}_1 + \boldsymbol{\kappa}_2$ og $\boldsymbol{\mu}_2 = \boldsymbol{\lambda}_2 - \boldsymbol{\kappa}_1$.

Vi tar for oss Lorentz-transformasjonen

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha_1, \alpha_2, \alpha_3) = \exp(\alpha_1\boldsymbol{\mu}_1 + \alpha_2\boldsymbol{\mu}_2 + \alpha_3\boldsymbol{\lambda}_3) , \tag{3}$$

definert av tre reelle parametre $\alpha_1, \alpha_2, \alpha_3$.

Den spesielle firer-impulsen k^μ definert i ligning (2) er invariant under $\boldsymbol{\Lambda}$, det vil si at $\Lambda^\mu{}_\nu \, k^\nu = k^\mu$. Hvorfor?

b) Regn ut $\boldsymbol{\Lambda}(\alpha, 0, 0) = \exp(\alpha\boldsymbol{\mu}_1)$, og sjekk i dette spesialtilfellet at $\Lambda^\mu{}_\nu \, k^\nu = k^\mu$.

c) Ligning (3) definerer en tredimensjonal undergruppe av Lorentz-gruppen, som Wigner kalte den «lille gruppen» til firer-impulsen k^μ.

Finn kommutasjonsrelasjonene til de tre generatorene $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\lambda}_3$ for denne gruppen.

d) I en unitær representasjon av den lille gruppen representerer vi

$$\boldsymbol{\mu}_1 \mapsto -\mathrm{i}M_1 \ , \qquad \boldsymbol{\mu}_2 \mapsto -\mathrm{i}M_2 \ , \qquad \boldsymbol{\lambda}_3 \mapsto -\mathrm{i}S_3 \ ,$$

der M_1, M_2, S_3 er Hermitiske operatorer.

Definer $M_\pm = M_1 \pm \mathrm{i}M_2$, og vis kommutasjonsrelasjonene

$$[S_3, M_+] = M_+ \ , \qquad [S_3, M_-] = -M_- \ .$$

Bruk disse to relasjonene til å vise at hvis $|\sigma\rangle$ er en egenvektor til S_3 med egenverdi σ, det vil si at

$$S_3 \,|\sigma\rangle = \sigma \,|\sigma\rangle \ ,$$

og hvis $M_+|\sigma\rangle \neq 0$, $M_-|\sigma\rangle \neq 0$, så er $M_+|\sigma\rangle$ en egenvektor til S_3 med egenverdi $\sigma + 1$, og $M_-|\sigma\rangle$ er en egenvektor til S_3 med egenverdi $\sigma - 1$.

e) Vis at $M_1^2 + M_2^2 = M_+M_- = M_-M_+$ er en Casimir-operator.
Det vil si at den kommuterer med alle tre operatorene M_1, M_2, S_3.

Det følger at $M_1^2 + M_2^2$ og S_3 har felles egenvektorer $|\rho, \sigma\rangle$ slik at

$$(M_1^2 + M_2^2)\,|\rho, \sigma\rangle = \rho^2\,|\rho, \sigma\rangle \ , \qquad S_3\,|\rho, \sigma\rangle = \sigma\,|\rho, \sigma\rangle \ .$$

Hvilke verdier kan ρ og σ ta i en generell irredusibel unitær representasjon av den lille gruppen?

Hvordan representeres M_+ og M_-?

Fotoner er masseløse partikler, og en kvantetilstand til et foton med firer-impuls k^μ transformeres ved en irredusibel representasjon av denne lille gruppen.

Hvilke verdier tar ρ og σ for fotonet?

f) En paritetstransformasjon som lar $k^\mu = (k, 0, 0, k)$ invariant er, for eksempel,

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Beregn den adjungerte virkningen av $\mathbf{Q}$, definert som $\boldsymbol{\lambda}_i \mapsto \mathbf{Q}\boldsymbol{\lambda}_i\mathbf{Q}^{-1}$, $\boldsymbol{\kappa}_i \mapsto \mathbf{Q}\boldsymbol{\kappa}_i\mathbf{Q}^{-1}$.

I kvanteteorien representeres $\mathbf{Q}$ som en unitær operator Q. Hva er $Q\,|\rho, \sigma\rangle$?

Department of physics

Examination paper for FY8104 / FY3105 Symmetry groups in physics

Academic contact during examination: Jan Myrheim

Phone: 73 59 36 53 / 900 75 172

Examination date: December 15, 2015

Examination time: 9–13

Permitted support material: Calculator, mathematical and physical tables

Language: English

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Examination, course FY8104/FY3105 Symmetry in physics
Tuesday December 15, 2015
Time: 09:00–13:00

Grades made public: Friday January 15, 2016

Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

Problem 1:

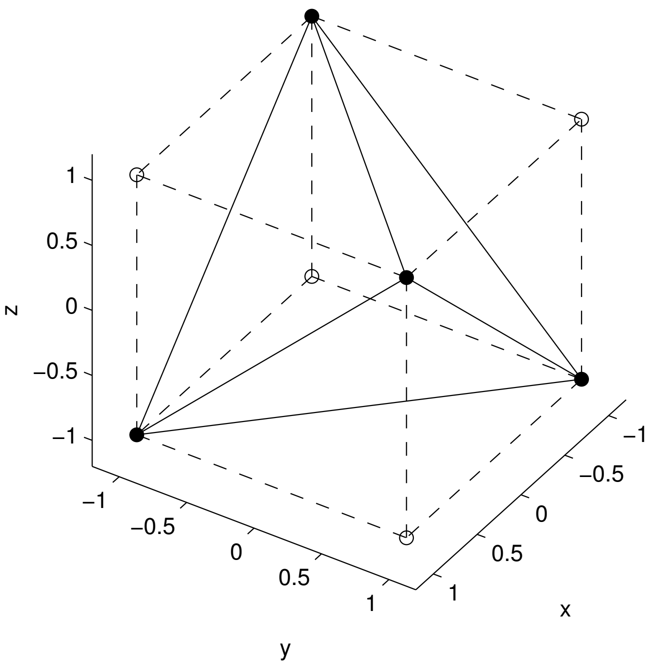

Figure 1: The corners of a regular tetrahedron are half the corners of a cube.

The eight points $(x, y, z) = (\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$ are the corners of a cube. The subset of points with $xyz = 1$ are the corners of a regular tetrahedron. See Figure 1.

The corners of the tetrahedron are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} , \quad \mathbf{c} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} , \quad \mathbf{d} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} . \tag{1}$$

The other corners of the cube are $-\mathbf{a}$, $-\mathbf{b}$, $-\mathbf{c}$, $-\mathbf{d}$, they are the corners of another tetrahedron.

a) The matrix

$$\mathbf{M}_{\mathrm{ab}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

belongs to the Lie group O(3), but not to SO(3). Explain.

It transforms the tetrahedron as follows,

$$\mathbf{M}_{\mathrm{ab}}\mathbf{a} = \mathbf{b} , \qquad \mathbf{M}_{\mathrm{ab}}\mathbf{b} = \mathbf{a} , \qquad \mathbf{M}_{\mathrm{ab}}\mathbf{c} = \mathbf{c} , \qquad \mathbf{M}_{\mathrm{ab}}\mathbf{d} = \mathbf{d} .$$

What is the geometrical interpretation, in terms of rotations and reflections?

The notation indicates that $\mathbf{M}_{\mathrm{ab}}$ interchanges the corners $\mathbf{a}$ and $\mathbf{b}$ of the tetrahedron. It is one of six transformation matrices that interchange two corners without moving the two other corners. Find the other five matrices.

b) The matrices $\mathbf{M}_{\mathrm{abc}} = \mathbf{M}_{\mathrm{ab}}\mathbf{M}_{\mathrm{bc}}$ and $\mathbf{M}_{\mathrm{abcd}} = \mathbf{M}_{\mathrm{ab}}\mathbf{M}_{\mathrm{bc}}\mathbf{M}_{\mathrm{cd}}$ also represent symmetry transformations of the tetrahedron.

How do these two transformations permute the corners of the tetrahedron?

How can they be interpreted geometrically, as rotations and reflections?

c) The symmetry group of the tetrahedron consists of all permutations of the four corners, therefore it is isomorphic to the symmetric group S_4.

We have found here a three dimensional representation of S_4. Is it irreducible?

Explain how your answer follows from the character table of S_4, given here.

	1	3	8	6	6
	1^4	2^2	31	21^2	4
χ_1	1	1	1	1	1
χ_2	3	−1	0	1	−1
χ_3	2	2	−1	0	0
χ_4	3	−1	0	−1	1
χ_5	1	1	1	−1	−1

Explain the notation used in the table for the conjugation classes of S_4.

The number of elements in each conjugation class is given in the table.

d) Consider now the symmetry group of the cube.

What is the order (the number of elements) of this group?

How can you enlarge the symmetry group of the tetrahedron to get the symmetry group of the cube? The answer may be easier than you think.

Write down the character table of the symmetry group of the cube.

Problem 2:

Methane, CH_4, is a powerful greenhouse gas because it absorbs infrared radiation with frequencies that excite vibrations of the molecule.

A molecule of methane in its ground state has the shape of a regular tetrahedron. Assume that the carbon atom is located at the origin and the four hydrogen atoms at the positions $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as defined in Equation (1) and in Figure 1.

Let, for example, $\mathbf{R}$ be the symmetry transformation of the tetrahedron such that $\mathbf{Ra} = \mathbf{b}$, $\mathbf{Rb} = \mathbf{c}$, $\mathbf{Rc} = \mathbf{a}$, $\mathbf{Rd} = \mathbf{d}$. Since the hydrogen atoms are identical, from the physical point of view this is an identity transformation of the molecule in its ground state.

Consider a small deformation of the molecule in which the hydrogen atom at $\mathbf{x}$ moves to $\mathbf{x} + \Delta\mathbf{x}$, where $\mathbf{x} = \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$. Then $\mathbf{R}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{Rx} + \mathbf{R}\Delta\mathbf{x}$. Since, for example, $\mathbf{Rc} = \mathbf{a}$, and the hydrogen atoms are identical, this is the same physical transformation as

$$\begin{pmatrix} \Delta\mathbf{a} \\ \Delta\mathbf{b} \\ \Delta\mathbf{c} \\ \Delta\mathbf{d} \end{pmatrix} \overset{\mathbf{R}}{\longmapsto} \begin{pmatrix} \mathbf{R}\Delta\mathbf{c} \\ \mathbf{R}\Delta\mathbf{a} \\ \mathbf{R}\Delta\mathbf{b} \\ \mathbf{R}\Delta\mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a} \\ \Delta\mathbf{b} \\ \Delta\mathbf{c} \\ \Delta\mathbf{d} \end{pmatrix} .$$

Thus, we represent the 3×3 matrix $\mathbf{R}$ by the 12×12 matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{R} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbf{R} .$$

In summary, a permutation $p \in S_4$ is represented by a 4×4 matrix $\mathbf{D}_\alpha(p)$, a 3×3 matrix $\mathbf{D}_\beta(p)$, and a 12×12 matrix $\mathbf{D}(p) = \mathbf{D}_\alpha(p) \otimes \mathbf{D}_\beta(p)$. The three transpositions $T_1 = (12)$, $T_2 = (23)$, and $T_3 = (34)$ generate S_4 and are represented as follows,

$$\mathbf{D}_\alpha(T_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\alpha(T_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\alpha(T_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{D}_\beta(T_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{D}_\beta(T_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{D}_\beta(T_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

a) How do the representations $\mathbf{D}_\alpha$, $\mathbf{D}_\beta$, and $\mathbf{D} = \mathbf{D}_\alpha \otimes \mathbf{D}_\beta$ of S_4 decompose into irreducible representations?

The character table of S_4 is given under problem 1c).

b) The methane molecule has three rotation modes and nine vibration modes in which the hydrogen atoms move but the centre of mass does not move. It has also three translation modes, not considered here, in which the carbon atom and the four hydrogen atoms move together.

How many *different* vibration frequencies does it have?
Explain briefly your reasoning.

Problem 3:

In this problem we use "natural units" where $c = 1$ and $\hbar = 1$. The matrices

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\boldsymbol{\kappa}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\kappa}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{\kappa}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are generators of Lorentz transformations, including rotations.
They satisfy the commutation relations

$$[\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j] = \epsilon_{ijk}\boldsymbol{\lambda}_k \;, \qquad [\boldsymbol{\lambda}_i, \boldsymbol{\kappa}_j] = \epsilon_{ijk}\boldsymbol{\kappa}_k \;, \qquad [\boldsymbol{\kappa}_i, \boldsymbol{\kappa}_j] = -\epsilon_{ijk}\boldsymbol{\lambda}_k \;.$$

Quantum states of a particle of mass $m > 0$ and spin s can be generated by Lorentz transformations from states where the particle is at rest.

A particle of mass $m = 0$ has no rest frame, and we have to choose other reference states than the states at rest, for example states where the particle moves in the positive z direction with four-momentum

$$k^\mu = (k^0, k^1, k^2, k^3) = (k, 0, 0, k) \;, \qquad k > 0 \;. \tag{2}$$

We obtain more general states by Lorentz transforming these reference states.

a) Define $\boldsymbol{\mu}_1 = \boldsymbol{\lambda}_1 + \boldsymbol{\kappa}_2$ and $\boldsymbol{\mu}_2 = \boldsymbol{\lambda}_2 - \boldsymbol{\kappa}_1$.

Consider the Lorentz transformation

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha_1, \alpha_2, \alpha_3) = \exp(\alpha_1\boldsymbol{\mu}_1 + \alpha_2\boldsymbol{\mu}_2 + \alpha_3\boldsymbol{\lambda}_3) \;, \tag{3}$$

defined by three real parameters $\alpha_1, \alpha_2, \alpha_3$.

The special four-momentum k^μ defined in Equation (2) is invariant under $\boldsymbol{\Lambda}$, that is, $\Lambda^\mu_{\ \nu}\, k^\nu = k^\mu$. Why?

b) Compute $\boldsymbol{\Lambda}(\alpha, 0, 0) = \exp(\alpha\boldsymbol{\mu}_1)$, and check in this special case that $\Lambda^\mu_{\ \nu}\, k^\nu = k^\mu$.

c) Equation (3) defines a three dimensional subgroup of the Lorentz group, called by Wigner the "little group" of the four-momentum k^μ.

Find the commutation relations of the three generators $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\lambda}_3$ of this group.

d) In a unitary representation of the little group we represent

$$\boldsymbol{\mu}_1 \mapsto -\mathrm{i}M_1 \ , \qquad \boldsymbol{\mu}_2 \mapsto -\mathrm{i}M_2 \ , \qquad \boldsymbol{\lambda}_3 \mapsto -\mathrm{i}S_3 \ ,$$

where M_1, M_2, S_3 are Hermitian operators.

Define $M_\pm = M_1 \pm \mathrm{i}M_2$, and prove the commutation relations

$$[S_3, M_+] = M_+ \ , \qquad [S_3, M_-] = -M_- \ .$$

Use these two relations to show that if $|\sigma\rangle$ is an eigenvector of S_3 with eigenvalue σ, that is,

$$S_3\,|\sigma\rangle = \sigma\,|\sigma\rangle \ ,$$

and if $M_+|\sigma\rangle \neq 0$, $M_-|\sigma\rangle \neq 0$, then $M_+|\sigma\rangle$ is an eigenvector of S_3 with eigenvalue $\sigma + 1$, and $M_-|\sigma\rangle$ is an eigenvector of S_3 with eigenvalue $\sigma - 1$.

e) Show that $M_1^2 + M_2^2 = M_+M_- = M_-M_+$ is a Casimir operator.
That is, it commutes with all three of M_1, M_2, S_3.

It follows that $M_1^2 + M_2^2$ and S_3 have common eigenvectors $|\rho, \sigma\rangle$ such that

$$(M_1^2 + M_2^2)\,|\rho, \sigma\rangle = \rho^2\,|\rho, \sigma\rangle \ , \qquad S_3\,|\rho, \sigma\rangle = \sigma\,|\rho, \sigma\rangle \ .$$

What values can ρ and σ take in a general irreducible unitary representation of the little group?

How are M_+ and M_- represented?

Photons are massless particles, and a quantum state of a photon with four-momentum k^μ transforms according to an irreducible representation of this little group.

What values do ρ and σ take for the photon?

f) A parity transformation leaving $k^\mu = (k, 0, 0, k)$ invariant is, for example,

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Compute the adjoint action of $\mathbf{Q}$, defined as $\boldsymbol{\lambda}_i \mapsto \mathbf{Q}\boldsymbol{\lambda}_i\mathbf{Q}^{-1}$, $\boldsymbol{\kappa}_i \mapsto \mathbf{Q}\boldsymbol{\kappa}_i\mathbf{Q}^{-1}$.

In the quantum theory $\mathbf{Q}$ is represented as a unitary operator Q. What is $Q\,|\rho, \sigma\rangle$?