TFY4230 Statistisk fysikk.

Kont. eksamen 12.08.2017

Løsningsforslag

1a)

$$\beta p V = \ln Z_g$$
$$= -\sum_k \ln\left(1 - e^{-\beta(\varepsilon_k - \mu)}\right)$$
$$= \sum_k f(\varepsilon_k)$$

$$f(\varepsilon_k) = -\ln\left(1 - e^{-\beta(\varepsilon_k-\mu)}\right)$$

$$\sum_k f(\varepsilon_k) = \sum_k \int_{-\infty}^{\infty} d\varepsilon\, \delta(\varepsilon - \varepsilon_k)\, f(\varepsilon)$$
$$= \int_{-\infty} d\varepsilon\, g(\varepsilon)\, f(\varepsilon)$$

$$g(\varepsilon) \equiv \sum_k \delta(\varepsilon - \varepsilon_k)$$

$\Rightarrow$

$$\underline{\underline{\beta p V = -\int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon) \ln\left(1 - e^{-\beta(\varepsilon-\mu)}\right)}}$$

$$\langle N \rangle = \frac{\partial(\ln Z_g)}{\partial \beta\mu}$$
$$= -\frac{\partial}{\partial \beta\mu} \sum_k \ln\left(1 - e^{-\beta(\varepsilon_k-\mu)}\right)$$
$$= \sum_k \frac{e^{-\beta(\varepsilon_k-\mu)}}{1 - e^{-\beta(\varepsilon_k-\mu)}}$$

$$= \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

Here, we have to be careful when we convert the sum to an integral, in that we have to single out the term containing the lowest value of ε_k, bearing in mind that

$$\mu < (\varepsilon_k)_{min}.$$

Hence,

$$\langle N \rangle = N_0 + \int_{-\infty}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon - \mu)} - 1}$$

where $N_0 \equiv \dfrac{1}{e^{\beta(\varepsilon_0 - \mu)} - 1}$

$$\varepsilon_0 \equiv (\varepsilon_k)_{min}$$

$$\langle N \rangle = \sum_k \langle n_k \rangle \Rightarrow \langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$H = \sum_k \varepsilon_k n_k$$

$$U = \langle H \rangle = \sum_k \varepsilon_k \langle n_k \rangle$$

$$= \sum_k \frac{\varepsilon_k}{e^{\beta(\varepsilon_k - \mu)} - 1}$$

$$= \int_{-\infty}^{\infty} d\varepsilon \, \frac{g(\varepsilon)\, \varepsilon}{e^{\beta(\varepsilon-\mu)} - 1}$$

Note that in this case, we have no additional separate term involving ε_0, since $\varepsilon_0 = 0$.

b) $$\beta p V = -\int_{-\infty}^{\infty} d\varepsilon \, g(\varepsilon) \ln\left(1 - e^{-\beta(\varepsilon-\mu)}\right)$$

We transform this expression by a partial integration, to obtain

$$\beta p V = \int_{-\infty}^{\infty} d\varepsilon \, \frac{G(\varepsilon)\, \beta}{e^{\beta(\varepsilon-\mu)} - 1}$$ NB!!

$$G'(\varepsilon) = g(\varepsilon)$$

Here, $g(\varepsilon) = A \; \varepsilon \; \Theta(\varepsilon)$

$$G(\varepsilon) = \frac{1}{2} g(\varepsilon) \, \varepsilon$$

Hence

$$pV = \frac{1}{2} \int_{-\infty}^{\infty} d\varepsilon \; \frac{g(\varepsilon)\,\varepsilon}{e^{\beta(\varepsilon-\mu)}-1} = \underline{\frac{U}{2}}$$

$$\underline{\underline{K=2}}$$

<u>c)</u>

$$N_0 = \langle N \rangle - \int_{-\infty}^{\infty} d\varepsilon \; \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}$$

At $T=0$, $N_0 = \langle N \rangle$, all particles are in the lowest-energy state. As T increases, N_0 is reduced, eventually reaching the value $N_0 = 0$ as $T \to T_\lambda$. Thus, at $T = T_\lambda$, we have:

$$\langle N \rangle = \int_{-\infty}^{\infty} d\varepsilon \; \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}$$

$$= V \frac{2\pi}{(hc)^2} \int_0^{\infty} d\varepsilon \; \frac{\varepsilon}{e^{\beta\varepsilon}-1}$$

$$\int_0^\infty d\varepsilon \, \frac{\varepsilon}{e^{\beta\varepsilon}-1} = \frac{1}{\beta^2} \int_0^\infty dx \, \frac{x}{e^x - 1}$$

$$= \frac{1}{\beta^2} \zeta(2)\Gamma(2)$$

$$\rho = \frac{2\pi}{(\beta h c)^2} \zeta(2)\Gamma(2)$$

We may now define a lengthscale in the problem, an ultrarelativistic version of the thermal wavelength

$$\lambda_T \equiv \beta c$$

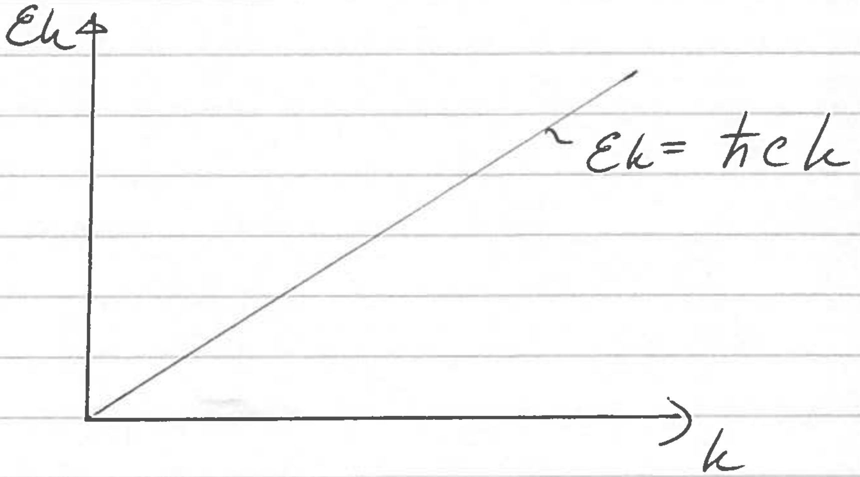
Then the BEC criterion is given by:

$$\underline{\lambda_T^2 \rho = 2\pi \zeta(2)\Gamma(2) \sim \mathcal{O}(1)}$$

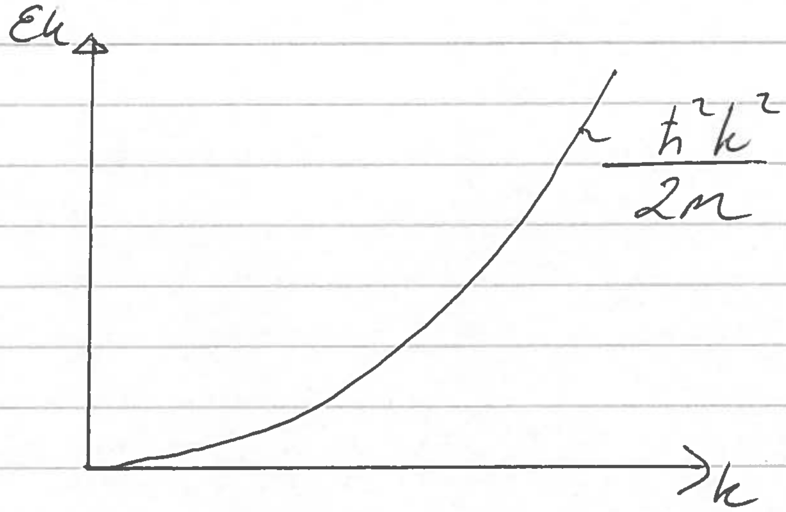
$$(k_B T_\lambda)^2 = \frac{1}{2\pi\zeta(2)\Gamma(2)} \rho (hc)^2$$

$$\underline{k_B T_\lambda = \frac{\rho^{1/2}}{\sqrt{2\pi\zeta(2)\Gamma(2)}} hc}$$

d) Ultrarel-case:



Non-rel. case:



The energy required to excite particles out of the ground state is ε_k. In 2D, the DOS $g(\varepsilon) \sim \varepsilon$ for ultrarel. particles and $g(\varepsilon) \sim$ constant for non-rel. particles. Thus, there are infinitely more states available at low energies in the non-rel. case compared to the ultra-rel. case. This is the origin of the difference in the BEC-behavior at low T the two cases.

2a

$$Z = Z_1^N \qquad \text{Independent spins}$$

$$Z_1 = \sum_{\sigma_i = \pm 1, 0} e^{\beta h S \sigma_i}$$

$$= 1 + e^{\beta h S} + e^{-\beta h S}$$

$$= 1 + 2\cosh(\beta h S)$$

$$\underline{\underline{Z = \left(1 + 2\cosh(\beta h S)\right)^N}}$$

b) $$H_e = -\frac{\partial \ln Z}{\partial \beta} = -N \frac{2hS\sinh(\beta h S)}{1+2\cosh(\beta h S)}$$

$$\underline{\underline{= -2NhS \frac{\sinh x}{1+2\cosh x}}} \; ; \; x \equiv \beta h S$$

2c

$$C_h = \left(\frac{\partial H_e}{\partial T}\right)_h = -\frac{1}{k_B T^2}\frac{\partial H_e}{\partial \beta}$$

$$H_e = -\frac{\partial \ln Z}{\partial \beta} = k_B T^2 \frac{\partial \ln Z}{\partial T}$$

$$C_h = \frac{1}{k_B T^2}\frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$= k_B \beta^2 \frac{\partial^2 \ln Z}{\partial \beta^2}$$

$$= k_B \beta^2 \frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)$$

$$= k_B \beta^2 \left[\frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)^2\right]$$

$$\left.\begin{aligned} \frac{1}{Z}\frac{\partial Z}{\partial \beta} &= -\langle H \rangle \\ \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2} &= \langle H^2 \rangle \end{aligned}\right\} \quad \underline{\underline{C_h = k_B \beta^2 \left[\langle H^2 \rangle - \langle H \rangle^2\right]}}$$

Q. E. D.

2d) The simplest way of doing this is to use the basic expression

$$C_h = \left(\frac{\partial H_e}{\partial T}\right)_h = -k_B \beta^2 \frac{\partial H_e}{\partial \beta}$$

and then use an approximation for H_e valid when $\beta h S \gg 1$

$$H_e = -2NhS \frac{\sinh x}{1 + 2\cosh x}$$

$$\simeq -NhS \tanh x$$

For large x, $\tanh x$ approaches 1 from below, but since we are going to take a derivative to obtain C_h, we need to know precisely how $\tanh x$ approaches 1.

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} \simeq 1 - 2e^{-2x}$$

$$C_h = -k_B \beta^2 hS \frac{\partial H_e}{\partial x}$$

$$= k_B \beta^2 (hS)^2 N \, 4 \, e^{-2\beta hS}$$

$$\underline{\underline{= N k_B (2\beta hS)^2 e^{-2\beta hS}}}$$

The heat capacity vanishes essentially exponentially as T is lowered to very low T ($\beta h S \gg 1$).

Physical explanation: Heat capacity originates with <u>energy-fluctuations</u> (problem 2c).

Energy-fluctuations originate with <u>spin-fluctuations</u> in this model

In this generalized Ising-model, spins can take three discrete values $\sigma_i = (0, \pm 1)S$. The energy difference between each state is hS, and this roughly the minimum amount of thermal energy $k_B T$ that must be supplied in order to flip a spin. This energy is not available when $k_B T \ll hS$ or equivalently $\beta h S \gg 1$. Thus when $k_B T \ll hS$, there are no spin-fluctuations $\Rightarrow$ no energy-fluctuations $\Rightarrow$ $C_h \to 0$ (exponentially)

3a

$$U = \langle H \rangle = \frac{\frac{1}{N!\, h^{2N}} \int \cdots \int d\Gamma \, H \, e^{-\beta H}}{Z}$$

$$= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \underline{\underline{-\frac{\partial \ln Z}{\partial \beta}}}$$

b)

$$Z = \frac{1}{N!}\frac{1}{h^{2N}}\left(Z_{1p}\right)^N \left(Z_{1r}\right)^N$$

$$Z_{1p} \equiv \int d^2p \, e^{-\frac{\beta p^2}{2m}}$$

$$= 2\pi \int_0^\infty dp \, p \, e^{-\frac{\beta p^2}{2m}}$$

$$= \pi \int_0^\infty dx \, e^{-\frac{\beta}{2m} x}$$

$$= \underline{\frac{2\pi m}{\beta}} = 2\pi m k_B T$$

$$Z_{1r} \equiv \int_V d^2r \, e^{-\beta V_0 r^2} = Q_1$$

$$= 2\pi \int_0^R dr \, r \, e^{-\frac{r^2}{R_0^2}} \; ; \quad \beta V_0 \equiv \frac{1}{R_0^2}$$

Thus, we have:

$$Z = \frac{1}{N!} \frac{1}{h^{2N}} \left(2\pi m k_B T\right)^N Q_1^N$$

$$= \frac{1}{\lambda^{2N} N!} Q_1^N \; ; \; \lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}$$

c) Explicit computation of Q_1:

$$Q_1 = \pi \int_0^{R^2} dx \, e^{-\frac{x}{R_0^2}}$$

$$= \pi R_0^2 \int_0^{\frac{R^2}{R_0^2}} du \, e^{-u}$$

$$= \pi R_0^2 \left(1 - e^{-\frac{R^2}{R_0^2}}\right)$$

$$p = k_B T \left(\frac{\partial \ln Z}{\partial V}\right)_T = N k_B T \left(\frac{\partial \ln Q_1}{\partial V}\right)_T$$

$$= \frac{N k_B T}{\pi} \frac{\partial \ln Q_1}{\partial R^2}$$

$$p = \frac{N k_B T}{\pi} \frac{\partial}{\partial R^2} \ln\left(1 - e^{-\frac{R^2}{R_0^2}}\right)$$

$$= \frac{\frac{N k_B T}{\pi R_0^2} \frac{1}{e^{\frac{R^2}{R_0^2}}}}{e^{\frac{R^2}{R_0^2}} - 1}$$

$$\frac{R}{R_0} << 1 \Rightarrow e^{\frac{R^2}{R_0^2}} - 1 \simeq \frac{R^2}{R_0^2}$$

$$\underline{\underline{p = \frac{N k_B T}{\pi R^2} = \frac{N k_B T}{V}}}$$ Ideal gas law

$$\frac{R}{R_0} >> 1 \Rightarrow e^{\frac{R^2}{R_0^2}} - 1 \simeq e^{\frac{R^2}{R_0^2}}$$

$$\underline{\underline{p = \frac{N k_B T}{\pi R_0^2} e^{-\frac{R^2}{R_0^2}}}}$$

d) The total internal energy is given by

$$U = -\frac{\partial \ln Z}{\partial \beta}$$

$$= -N\frac{\partial \ln Z_{cp}}{\partial \beta} - N\frac{\partial \ln Q_i}{\partial \beta}$$

$$= Nk_BT + Nk_BT - N\frac{\partial}{\partial \beta}\ln\left(1 - e^{-\frac{R^2}{R_0^2}}\right)$$

$$= 2Nk_BT - \frac{N\,V_0 R^2}{e^{\frac{R^2}{R_0^2}} - 1}$$

$$= \langle H \rangle$$

$$= N\left\langle \frac{p_{ri}^2}{2M} + \frac{p_{\theta i}^2}{2mr_i^2} + V_0 r_i^2 \right\rangle$$

$$= N\left[\frac{k_BT}{2} + \left\langle \frac{p_{\theta i}^2}{2mr_i^2} + V_0 r_i^2 \right\rangle\right]$$

$$\left\langle \frac{p_{\theta i}^2}{2mr_i^2} + V_0 r_i^2 \right\rangle = \frac{3Nk_BT}{2} - \frac{N\,V_0 R^2}{e^{\frac{R^2}{R_0^2}} - 1}$$