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1a) $$H = -J\sum_{\langle i,j\rangle} \sigma_i \sigma_j - B\sum_i \sigma_i$$

$$\sigma_i = m + \delta\sigma_i \Rightarrow \langle \sigma_i \rangle = m$$
$$\delta\sigma_i = \sigma_i - m$$

$$H = -J\sum_{\langle i,j\rangle} (m+\delta\sigma_i)(m+\delta\sigma_j) - B\sum_i (m+\delta\sigma_i)$$

$$\simeq -Jm^2Nz - Jmz\sum_i \delta\sigma_i - Jmz\sum_j \delta\sigma_j$$
$$- BmN - B\sum_i \delta\sigma_i$$

$$= -Jm^2Nz + 2Jm^2Nz - 2Jmz\sum_i \sigma_i - B\sum_i \sigma_i$$

$$= Jm^2Nz - (B+2Jmz)\sum_i \sigma_i$$

$$= \underline{\underline{Jm^2Nz - B_{eff}\sum_i \sigma_i}} \quad \text{q.e.d.} \quad (*)$$

b) $$Z = \sum_{\{\sigma_i\}} e^{-\beta H} = e^{-\beta G}$$

$$= \left[\sum_{\sigma_1} e^{-\beta(Jm^2 z - B_{eff}\sigma_1)}\right]^N$$

$$= Z_1^N$$

(*) B_{eff} is an effective magnetic field felt by each spin from external B and surrounding spins. m is the magnetization of the system

$$Z_1 = \sum_{\sigma=0,\pm1,\pm2} e^{-\beta J m^2 z + \beta B_{eff}\sigma}$$

$$= e^{-\beta J m^2 z}\left(1 + 2\cosh\Omega + 2\cosh 2\Omega\right)$$

$$\Omega \equiv \beta B_{eff} = \beta(B + 2Jzm) = \omega + \alpha m$$

$$\omega \equiv \beta B \; ; \; \alpha \equiv 2Jz\beta$$

Define $Y(\omega+\alpha m) = 2(\cosh\Omega + \cosh 2\Omega)$

$$Z = e^{-\beta G} = e^{-\beta J m^2 z N}(1+Y)^N$$

$$= e^{-\beta J m^2 z N + N\ln(1+Y)} \Rightarrow$$

$$G = Jm^2 zN - \frac{N}{\beta}\ln(1+Y) \; ; \; \text{q.e.d.}$$

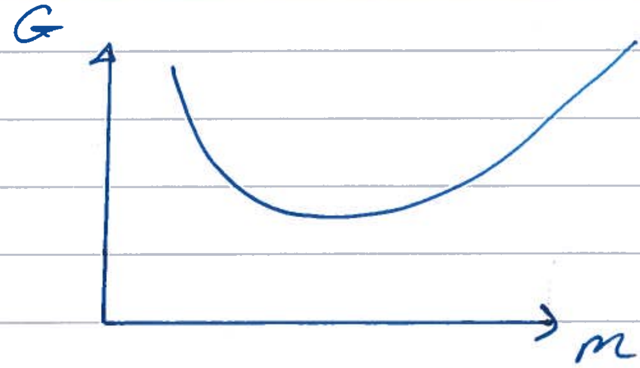
$$Y = Y(\omega + \alpha m) = Y(\beta B + 2\beta J z m)$$

Consider H: $B \to -B$; $\sigma_i \to -\sigma_i$

$\Rightarrow$ $m \to -m$ $\Rightarrow$ $\omega + \alpha m \to -(\omega+\alpha m)$

Under this transformation, H is invariant (does not change). It corresponds to „flipping" the system upside-down. This does not change the physics, when the field B is also „flipped" upside-down. Thus, G, and hence Y is invariant when $x \to -x$.

c) m is a so far unknown parameter that must be determined self-consistently by choosing the m that minimizes G.



$$(*) \quad \boxed{\frac{\partial G}{\partial m} = 0}$$

$$\frac{\partial G}{\partial m} = \frac{\partial}{\partial m}\left[J m^2 N z - \frac{N}{\beta} \ln\left[1 + Y(\omega + \alpha m)\right]\right]$$

$$= 2 J m z N - \frac{N}{\beta} \frac{1}{1+Y} \cdot \frac{\partial}{\partial m} Y(\omega + \alpha m)$$

$$= 2 J N z m - \frac{N}{\beta} \alpha \frac{Y'(\omega + \alpha m)}{1 + Y(\omega + \alpha m)}$$

Inserting the expression for α, dividing by $2JNz$, and using $(*)$, we find

$$m = \frac{Y'(\alpha + \omega m)}{1 + Y(\alpha + \omega m)}$$

d) T_c is defined only when $B=0$! If $B\neq 0$, there is no special T that separates the low-temp. phase from the high-temp. phase. So we set $B=0$. Then we get

$$m = \frac{Y'(\alpha m)}{1+Y(\alpha m)} \quad ; \quad \omega = 0$$

$$Y = 2\left(\cosh(\alpha m) + \cosh(2\alpha m)\right)$$

$$Y' = 2\left[\sinh(\alpha m) + 2\sinh(2\alpha m)\right]$$

$T \to T_c^-$: $m \ll 1 \Rightarrow$

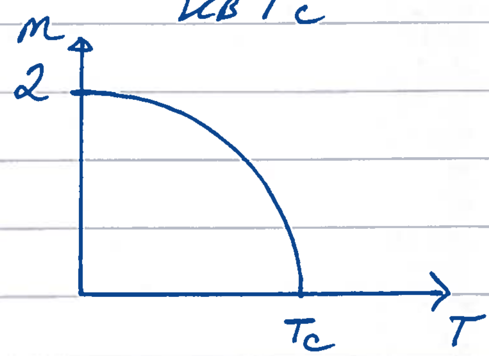
$$Y'(\alpha m) \simeq 2\left(\alpha m + 2\alpha m \cdot 2\right)$$

$$= 2\cdot 5 \cdot \alpha m \quad (\Rightarrow Y''(0) = 10)$$

$$m \simeq \frac{10\alpha m}{1+4} = 2\alpha m$$

$$1 = 2\cdot 2 J z \beta_c \quad ; \quad \beta_c = \frac{1}{k_B T_c}$$

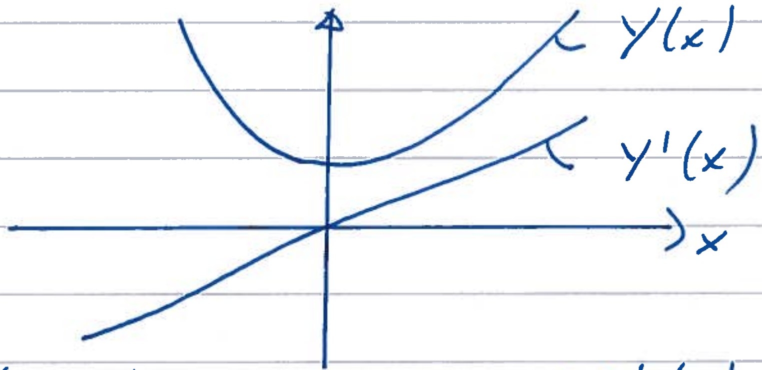
$$\underline{k_B T_c = 4 J z}$$



Even if we had not had an explicit expression for Y, we could still make progress.

$Y(0) = 4$

$Y(x) = Y(-x) \Rightarrow Y(x)$ is an even function of x, $\Rightarrow Y'(x)$ is an odd function of x.



$Y'(x=0) = 0$ and $Y'(x)$ must be linear in x when $|x| \ll 1$

$$Y'(x) \cong Y''(0)\, x \quad ; \quad x \ll 1$$

$$m \simeq \frac{Y''(0)}{5}\, \alpha m$$

$$1 = \frac{Y''(0)}{5}\, 2Jz\beta_c$$

$$\underline{k_B T_c = \frac{2Y''(0)}{5}\, Jz} \quad (= 4Jz)$$

2a)

$$Z_g = e^{\beta p V} \Rightarrow$$

$$\beta p V = \sum_k \ln\left(1 + e^{-\beta(\varepsilon_k - \mu)}\right)$$

$$= \int d\varepsilon \, g(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$g(\varepsilon) = \sum_k \delta(\varepsilon - \varepsilon_k)$$

$$\beta p V = V \frac{\Omega_d}{(\hbar c)^d} \int d\varepsilon \; \varepsilon^{d-1} \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$= \frac{V \Omega_d}{(hc)^d} \left\{ \left. \frac{1}{d} \varepsilon^d \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right) \right|_0^\infty - \frac{1}{d} \int_0^\infty d\varepsilon \; \varepsilon^d \frac{(-\beta) \cdot e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}} \right\}$$

$$\beta p V = V\beta \frac{\Omega_d}{(hc)^d} \frac{1}{d} \int_0^\infty d\varepsilon \frac{\varepsilon^d}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$\underline{\underline{p = \frac{\Omega_d}{(hc)^d} \frac{1}{d} \int_0^\infty d\varepsilon \frac{\varepsilon^d}{e^{\beta(\varepsilon-\mu)} + 1}}} \qquad \text{q.e.d.}$$

$$\langle N \rangle = \frac{\partial}{\partial(\beta\mu)} \ln Z_g$$

$$= \frac{\partial}{\partial(\beta\mu)} \int d\varepsilon\, g(\varepsilon) \ln\left(1 + e^{-\beta(\varepsilon-\mu)}\right)$$

$$= V \frac{\Omega_d}{(hc)^d} \int_0^\infty d\varepsilon\, \varepsilon^{d-1} \frac{e^{-\beta(\varepsilon-\mu)}}{1 + e^{-\beta(\varepsilon-\mu)}}$$

$$\underline{\underline{\rho = \frac{\Omega_d}{(hc)^d} \int_0^\infty d\varepsilon \frac{\varepsilon^{d-1}}{e^{\beta(\varepsilon-\mu)} + 1}}} \qquad \text{q.e.d.}$$

<u>b)</u>

$$p = \frac{\Omega_d}{(hc)^d} \frac{1}{d} \int_0^\infty d\varepsilon\, \varepsilon^d \sum_{l=1}^\infty (-1)^{l+1} e^{-\beta(\varepsilon-\mu)l}$$

$$= \frac{\Omega_d}{(hc)^d} \frac{1}{d} \sum_{l=1}^\infty (-1)^{l+1} z^l \underbrace{\int_0^\infty d\varepsilon\, \varepsilon^d e^{-\beta\varepsilon l}}$$

$$= \frac{1}{\beta^{d+1} l^{d+1}} \Gamma(d+1)$$

$$\beta p = \frac{\Omega_d}{(\beta hc)^d} \underbrace{\frac{\Gamma(d+1)}{d}}_{=\Gamma(d)} \sum_{l=1}^\infty \frac{(-1)^{l+1}}{l^{d+1}} z^l$$

$$\rho = \frac{\Omega d}{(hc)^d} \int_0^\infty d\varepsilon\, \varepsilon^{d-1} \sum_{l=1}^\infty (-1)^{l+1} e^{-\beta(\varepsilon-\mu)l}$$

$$= \frac{\Omega d}{(hc)^d} \sum_{l=1}^\infty (-1)^{l+1} z^l \underbrace{\int_0^\infty d\varepsilon\, \varepsilon^{d-1} e^{-\beta\varepsilon l}}_{= \frac{1}{\beta^d l^d} \Gamma(d)}$$

$$= \frac{\Omega d}{(\beta hc)^d} \Gamma(d) \sum_{l=1}^\infty \frac{(-1)^{l+1}}{l^d} z^l$$

Comparing with the given expression, we find

$$b_l = \frac{(-1)^{l+1}}{l^{d+1}}$$

$$\frac{1}{\lambda^d} = \frac{\Omega d\, \Gamma(d)}{(\beta hc)^d}$$

$$\lambda = \beta hc \left(\frac{1}{\Omega d\, \Gamma(d)} \right)^{1/d}$$

Alternatively, we could have defined

$$b_l = \frac{(-1)^{l+1}}{l^{d+1}} \Omega_d \Gamma(d)$$

$$\lambda = \beta h c$$

c)

$$\beta p = \frac{1}{\lambda^d} F(z)$$

$$\rho = \frac{1}{\lambda^d} G(z) = \frac{1}{\lambda^d} F'(z)$$

$$z = G^{-1}(\rho \lambda^d)$$

$$\beta p = \frac{1}{\lambda^d} F(G^{-1}(\rho \lambda^d))$$

$$= \frac{1}{\lambda^d} H(\rho \lambda^d)$$

$$\beta p = \frac{1}{\lambda^d}\left(z - \frac{z^2}{2^{1+d}} + \dots\right)$$

$$\rho = \frac{1}{\lambda^d}\left(z - \frac{z^2}{2^d} + \dots\right)$$

$$\beta p - \rho = \frac{1}{\lambda^d} z^2 \frac{1}{2^d}\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2^{d+1}} \frac{z^2}{\lambda^d}$$

$$z \simeq \rho \lambda^d$$

$$\beta p - \rho = \frac{1}{2^{d+1}} \rho^2 \lambda^d$$

$$\beta p = \rho + B_2 \rho^2 + \ldots$$

$$\underline{\underline{B_2 = \frac{1}{2^{d+1}} \lambda^d}}$$

$$\beta p = \rho \left(1 + \frac{1}{2^{d+1}} \lambda \rho^d + \ldots\right)$$

Parameter that controls the quantum corrections to ideal gas law: $\underline{\rho \lambda^d}$

$\rho \lambda^d$ becomes sizeable when the thermal wavelength becomes comparable to separation, r, between particles $\rho \sim \frac{1}{r^d}$

$\frac{\lambda}{r} \sim 1 \Rightarrow$ corrections become large.

Quantum corrections become important when wavefunctions of particles start to overlap.

Moreover, the positive correction to the pressure from the $B_2(T)\rho^2$-term means that, somehow, the overlap of wavefunctions of different particles represent a <u>repulsion</u> between them. Such „repulsion" between single-particle quantum states is characteristic of <u>fermionic</u> systems. This is what we would expect from the expression for Z_g:

$$Z_g = \prod_k \left(1 \pm e^{-\beta(\varepsilon_k - \mu)}\right)^{\pm 1}$$

+1: Fermions
−1: Bosons

3a) $$Z = \frac{1}{h^N}\frac{1}{N!}\left(Z_1\right)^N$$

$$Z_1 \equiv \int_{-\infty}^{\infty} dp\, e^{-\frac{\beta p^2}{2m}} \int_{-\infty}^{\infty} dx\, e^{-\beta\left(\frac{m\omega^2}{2}x^2+\alpha x^4\right)}$$

$$\approx \sqrt{\frac{2\pi m}{\beta}} \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta p^2}{2m}} \left(1-\beta\alpha x^4\right)$$

$$= \sqrt{\frac{2\pi m}{\beta}}\left(\sqrt{\frac{2\pi}{\beta m\omega^2}} - \beta\alpha\,\frac{3}{4}\sqrt{\pi}\,\frac{1}{\gamma^{5/2}}\right)$$

$$= \left(\frac{2\pi}{\beta\omega}\right)\left(1-\frac{3}{4}\frac{\beta\alpha}{\gamma^2}\right) \Longrightarrow$$

$$\underline{\underline{Z = \frac{1}{h^N}\frac{1}{N!}\left(\frac{2\pi}{\beta\omega}\right)^N\left(1-\frac{3}{4}\frac{\beta\alpha}{\gamma^2}\right)^N}}$$

This approximation is valid to linear order in α, i.e. when $1 \gg \frac{3}{4}\frac{\beta\alpha}{\gamma^2} \Leftrightarrow \underline{k_B T \ll \frac{m^2\omega^2}{\alpha}}$

b) $$U = \langle H \rangle = \frac{1}{Z}\frac{1}{h^N}\frac{1}{N!}\int d\Gamma\, H e^{-\beta H}$$

$$= -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \underline{-\frac{\partial \ln Z}{\partial \beta}}$$

$$U = -N\frac{\partial \ln(1/\beta)}{\partial\beta} - N\frac{\partial}{\partial\beta}\ln\left(1-\frac{3}{4}\frac{\beta\alpha}{\gamma^2}\right)$$

$$= Nk_BT - N\frac{1}{\left(1-\frac{3}{4}\frac{\beta\alpha}{\gamma^2}\right)}\cdot\left(-\frac{3}{4}\right)\frac{\alpha}{\left(\frac{m\omega^2}{2}\right)^2}\left(-\frac{1}{\beta^2}\right)$$

$$\cong \underline{\underline{Nk_BT\left(1-\frac{3}{4}\frac{\beta\alpha}{\gamma^2}\right)}};\quad \underline{\underline{\frac{3}{4}\frac{\beta\alpha}{\gamma^2} \ll 1}}$$

The result in 3a is reasonable when

$$\frac{3}{4}\frac{\beta\alpha}{\gamma^2} \ll 1$$

and the expansion of the exp-function is only valid to linear order in α.

$$U = Nk_BT - Nk_BT\,\frac{3}{4}\frac{\alpha}{\left(\frac{m\omega^2}{2}\right)^2}\,k_BT$$

$$\underline{\underline{C_V = Nk_B\left(1-\frac{3}{2}\frac{\beta\alpha}{\gamma^2}\right)}};\quad \frac{3}{2}\frac{\beta\alpha}{\gamma^2} \ll 1$$

c) $$U = \langle H \rangle = \langle E_{kin} + U_p \rangle$$

$\langle E_{kin} \rangle = \frac{N k_B T}{2}$ by the classical equipartition theorem

$$\langle U_p \rangle = U - \frac{N k_B T}{2}$$

$$= N k_B T \left(\frac{1}{2} - \frac{3}{4} \frac{\beta \alpha}{\gamma^2} \right)$$

Note that we cannot apply the generalized equipartition theorem to $\langle U_p \rangle$ itself, since the coordinate x_i appears in two places in the Hamiltonian, namely $\frac{1}{2} m \omega^2 x_i^2$ and αx_i^4.