

Problem 1

Exam 14.12.2020

①

a) $m \dot{v} + k v = F_0 + R(t)$

b) $\gamma \equiv \frac{k}{m}$

$$\dot{v} + \gamma v = \frac{F_0}{m} + \frac{R(t)}{m}$$

$$\frac{d}{dt} (e^{\gamma t} v(t)) = \frac{1}{m} (F_0 + R) e^{\gamma t}$$

$$e^{\gamma t} v(t) - v_0 = \frac{F_0}{m} \int_0^t dt' e^{\gamma t'} + \frac{1}{m} \int_0^t dt' R(t') e^{\gamma t'}$$

$$v(t) = v_0 e^{-\gamma t} + \frac{F_0}{m\gamma} (1 - e^{-\gamma t})$$

$$+ \frac{e^{-\gamma t}}{m} \int_0^t dt' R(t') e^{\gamma t'}$$

c) $\langle v(t) \rangle = v_0 e^{-\gamma t} + \frac{F_0}{m\gamma} (1 - e^{-\gamma t})$

$$\langle v^2(t) \rangle = \langle v(t) \rangle^2 + \frac{e^{-2\gamma t}}{m^2} \int_0^t dt_1 \int_0^t dt_2 e^{\gamma(t_1+t_2)} \langle R(t_1) R(t_2) \rangle$$

$$\langle R(t_1) R(t_2) \rangle = \langle R^2 \rangle \delta(t_1 - t_2) \quad (2)$$

Thus, the last integral becomes

$$I = \frac{e^{-2\gamma t}}{m^2} \langle R^2 \rangle \int_0^t \int_0^{t'} e^{2\gamma t'} dt' dt$$

$$= \frac{1}{2\gamma} \left(e^{2\gamma t} - 1 \right)$$

Hence

$$\sigma^2 = \langle (v - \langle v \rangle)^2 \rangle = I$$

$$= \frac{\langle R^2 \rangle}{2\gamma m^2} (1 - e^{-2\gamma t})$$

d) $t \rightarrow \infty$: Equilibrium

$$\frac{1}{2} m \sigma^2 = \frac{1}{2} k_B T \quad (\text{Equipartition thm.})$$

$$\sigma^2 = \frac{\langle R^2 \rangle}{2\gamma m^2} = \frac{k_B T}{m}$$

$$\langle R^2 \rangle = 2\gamma m k_B T$$

d) $\langle v(t_1) v(t_2) \rangle$

(e) $= \langle v(t_1) \rangle \langle v(t_2) \rangle$

$$+ \frac{e^{-\gamma(t_1+t_2)}}{m^2} \int_0^{t_1} \int_0^{t_2} e^{\gamma(t'+t'')} \langle R^2 \rangle \delta(t_1 - t_2) dt' dt''$$

Consider the final integral, I: ③

c) $\phi_1 > \phi_2$:

$$I = \frac{e^{-\gamma(\phi_1 + \phi_2)}}{m^2} \langle R^2 \rangle \int_0^{\phi_2} d\phi' e^{2\gamma\phi'}$$

$$= \frac{\langle R^2 \rangle}{2\gamma m^2} \left[e^{-\gamma(\phi_1 - \phi_2)} - e^{-\gamma(\phi_1 + \phi_2)} \right]$$

c) $\phi_2 > \phi_1$:

$$I = \frac{e^{-\gamma(\phi_1 + \phi_2)}}{m^2} \langle R^2 \rangle \int_0^{\phi_1} d\phi' e^{2\gamma\phi'}$$

$$= \frac{\langle R^2 \rangle}{2\gamma m^2} \left[e^{-\gamma(\phi_2 - \phi_1)} - e^{-\gamma(\phi_1 + \phi_2)} \right]$$

$$\phi_1 \rightarrow \infty, \quad \phi_2 \rightarrow \infty, \quad \phi_1 - \phi_2 = \tau$$

$$\langle v(\phi_1) \rangle \rightarrow \frac{F_0}{m\gamma}$$

$$\langle v(\phi_2) \rangle \rightarrow \frac{F_0}{m\gamma}$$

$$I \rightarrow \frac{\langle R^2 \rangle}{2\gamma m^2} e^{-\gamma|\tau|}$$

$$\langle v(\phi_1) v(\phi_2) \rangle \rightarrow \left(\frac{F_0}{m\gamma} \right)^2 + \frac{\langle R^2 \rangle}{2\gamma m^2} e^{-\gamma|\tau|}$$

(4)

$$f) \langle \langle v(t_1) v(t_2) \rangle \rangle$$

= autocorrelation function

$$= \langle v(t_1) v(t_2) \rangle - \langle v(t_1) \rangle \langle v(t_2) \rangle$$

$$= \frac{\langle R^2 \rangle}{2m^2 \gamma} \left[e^{-\gamma |t_1 - t_2|} - e^{-\gamma(t_1 + t_2)} \right]$$

$t_1, t_2 \rightarrow \infty$: Exponentially decaying with $|t_1 - t_2|$; short-time correlations. $v(t)$ is thus a Markov-process.

Note the importance of considering $\langle \langle v(t_1) v(t_2) \rangle \rangle$ instead of $\langle v(t_1) v(t_2) \rangle$ to address the question.

Alternatively, one may argue that $v(t)$ is determined uniquely by one initial condition.

Problem 2.

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$$a) \quad \frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll.}}$$

L. H. S.: A convective (total) derivative in $(t, \vec{r}, \vec{v})$ -space.

R. H. S.: A collision-term.

$\vec{v}$: Velocity

$\vec{a}$: Acceleration

$\vec{r}$: Position

t : time

f : Single-particle distribution-function

$$f = f(\vec{r}, \vec{v}, t)$$

$\left(\frac{\partial f}{\partial t} \right)_{\text{coll}}$: Term describing collisions between pairs of particles.

b) When specifying the collision-term $(\frac{d}{dt})_{\text{coll}}$ in terms of a product of two f 's for a pair of particles, a crucial assumption is that the colliding particles are statistically independent prior to (but not after) collision. This crucial assumption clearly breaks time-reversal symmetry. ⑥

c) H-theorem: Define a functional

$$H = \int d^3x \int d^3v f \ln f$$

Then $\frac{dH}{dt} \leq 0$

Thus, the Boltzmann-equation describes a temporal evolution of a system from an initial state towards an equilibrium state.

d) Boltzmann-equation
with $\vec{a} = 0$ and
 $f = f(t)$

L.H.S: $\frac{\partial f}{\partial t}$

R.H.S: $a(f - f_0)$

f_0 : Equilibrium solution

$$\frac{\partial f_0}{\partial t} = 0 \quad (\text{clearly})$$

$$\frac{\partial (f - f_0)}{\partial t} = a(f - f_0)$$

$$f = f_0 + \tilde{f} e^{at}$$

$$a < 0 \quad f_0? \quad f \rightarrow f_0 \quad (t \rightarrow \infty)$$

$$a = -1/\tau$$

τ : Relaxation time

$$f = f_0 + \tilde{f} e^{-t/\tau}$$

$$f_0: \lim_{t \rightarrow \infty} f(t)$$

$$\tilde{f} = f(0) - f_0$$