Norwegian University of Science and Technology
Department of Physics

Contact: Jacob Linder
Phone: 735 918 68

Suggested solution for Exam TFY4345: Classical Mechanics

NOTE: The solutions below are meant as guidelines for how the problems may be solved and do not necessarily contain all the detailed steps of the calculations.

PROBLEM 1

(a) Alternative ii).

(b) Alternative iii).

(c) Alternative ii).

(d) Alternative iv).

(e) Alternative iii).

(f) Alternative iv).

(g) From Hamilton's equations, we find that

$$\dot{q} = p/\alpha - bq\mathrm{e}^{-\alpha t} \tag{1}$$

so that $p = \alpha(\dot{q} + bq\mathrm{e}^{-\alpha t})$. The Lagrangian is given by $L = p\dot{q} - H$, which upon insertion and after cleaning up some terms gives:

$$L = \alpha\dot{q}^2/2 - kq^2/2 + b\alpha(q\dot{q}\mathrm{e}^{-\alpha t} - \alpha q^2\mathrm{e}^{-\alpha t}/2). \tag{2}$$

An equivalent Lagrangian is one that differs only by an extra term dF/dt where $F = F(q,t)$. We see that if $F(q,t) = b\alpha q^2\mathrm{e}^{-\alpha t}/2$, then the last term in the Hamiltonian above is exactly dF/dt. Thus, $L' = \alpha\dot{q}^2/2 - kq^2/2$ is an equivalent Lagrangian describing the same physics, and we may thus conclude that the original Hamiltonian describes a harmonic oscillator.

PROBLEM 2

(a) The potential energy is $V = k/(2r^2) = ku^2/2$ can be inserted into the equation for $\dot{r}$ stemming from conservation of energy:

$$\dot{r} = \sqrt{\frac{2}{m}[E - V - l^2/(2mr^2)]}. \tag{3}$$

We eliminate dt via $\dot{\theta} = l/(mr^2)$ and obtain:

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2mE}{l^2} - \left(1 + \frac{mk}{l^2}\right)u^2}} \tag{4}$$

upon integration and substituting $r = 1/u$. This integral can be solved analytically similarly to how we treated the Kepler problem in the lectures (can be looked up in Rottmann), and we find the solution:

$$u(\theta) = \sqrt{\frac{8mE}{l^2}\left(1 + \frac{mk}{l^2}\right)} \cos\left[\sqrt{1 + \frac{mk}{l^2}}(\theta - \theta_0)\right] \tag{5}$$

(b) The Lagrangian for the planet as it is in orbit reads $L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + k/r$ where the attractive Kepler-potential as usual is $V(r) = -k/r$. Note that the angular velocity $\dot{\theta}$ for the circular orbit is related to the period of the orbit τ via $\tau = 2\pi/\dot{\theta}$. The general equation of motion is

$$m\ddot{r} = \mu r\dot{\theta}^2 - k/r^2. \tag{6}$$

For a circular orbit, we find the radius of the orbit r_0 by setting $\ddot{r} = 0$ so that $r_0 = (k\tau^2/4\pi^2 m)^{1/3}$. Now, after we remove the planet's kinetic energy the equation of motion simplifies to:

$$m\ddot{r} = -k/r^2. \tag{7}$$

Multiplying this equation with $2\dot{r}$ we can rewrite it as:

$$\frac{d}{dt}(\dot{r}^2) = \frac{d}{dt}(2k/mr) \tag{8}$$

so that the solution is $\dot{r}^2 = 2k/mr + C$ where C is an integration constant. Since we know that $\dot{r} = 0$ at $t = t_0$, it follows that $C = -2k/mr_0$. Reinstating this into the equation, we find

$$\dot{r} = \sqrt{\frac{2k}{\mu}\left(\frac{1}{r} - \frac{1}{r_0}\right)}. \tag{9}$$

We can now evaluate the time required for the planet to fall a distance r_0 to the center of gravity. The time required is:

$$t_c = \int_{r_0}^{0} dr \cdot (dt/dr) = \int_{r_0}^{0} dr \cdot (dr/dt)^{-1} = \int_{r_0}^{0} dr(\dot{r})^{-1}. \tag{10}$$

Using our expression for $\dot{r}$ above and evaluating the integral, we arrive at

$$t_c = \frac{\tau}{4\sqrt{2}}. \tag{11}$$

(c) See discussion in compendium. Head-on collisions are favorable compared to firing at stationary targets in order to reduce the threshold energy.

(d) It was derived in the lectures (see also compendium section *Relativistic Kinematics* for details) that the threshold kinetic energy is given by (set $c = 1$):

$$K_\pi/m_\pi = M^2_{\text{final}} - M^2_{\text{before}}/(2m_\pi m_n) \tag{12}$$

where $M_{\text{final}} = m_K + m_\Lambda$ and $M_{\text{before}} = m_\pi + m_n$.

(e) There are two degrees of freedom since the length l is fixed: these are the angles describing the movement on a sphere, i.e. ϕ and θ. The kinetic and potential energies are:

$$T = ml^2\dot{\theta}^2/2 + ml^2\sin^2\theta\dot{\phi}^2/2,\ V = mgl\cos\theta. \tag{13}$$

Constructing the Lagrangian and deriving the equations of motion, we find:

$$\frac{d}{dt}\left(\dot{\phi}\sin\theta^2\right) = 0,\ \ddot{\theta} + \sin\theta(g/l - \cos\theta\dot{\phi}^2) = 0. \tag{14}$$

(f) Use the conservation of 4-momentum, in particular the third and fourth components, to obtain the following two equations relating momentum and energy before and after the collision:

$$\gamma_1 m_1 v_1 - \gamma_2 m_2 v_2 = \gamma_3 m_3 v_3 \tag{15}$$

$$\gamma_1 m_1 c + \gamma_2 m_2 c = \gamma_3 m_3 c. \tag{16}$$

The above equations may be manipulated to yield both the mass and the velocity of the produced particle:

$$m_3 = (\gamma_1/\gamma_3)m_1 + (\gamma_2/\gamma_3)m_2 \tag{17}$$

$$v_3 = \frac{\gamma_1 m_1 v_1 - \gamma_2 m_2 v_2}{\gamma_1 m_1 + \gamma_2 m_2}. \tag{18}$$

It follows from the equation for m_3 that we can only have $m_3 = 0$ and non-zero m_1 and m_2 if $\gamma_3 \to \infty$. To investigate if this is possible, we can find an expression for $\gamma_3^2 = 1/[1 - (v_3/c)^2]$:

$$\gamma_3^2 = \frac{(\gamma_1 m_1 + \gamma_2 m_2)^2}{m_1^2 + m_2^2 + 2\gamma_1\gamma_2 m_1 m_2(1 - v_1 v_2/c^2)}. \tag{19}$$

Since the denominator consists of three terms which all are greater than zero, we cannot obtain $\gamma_3 \to \infty$. as the denominator never goes to zero.

(g) See lecture notes and compendium for detailed discussion. For full score, the student must have elucidated the role of cyclic coordinates, canonical momenta, and the existence of a conservation law in the presence of a continuous symmetry.