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[17] This is one possible definition of the resonant particles. In Sec. IV we will have an alternative point of view, where particles in the (time-independent) resonant region, defined by Eq. (29), is considered as resonant.

[18] Even linearly the plane wave electric field will excite small contributions to $\tilde{f}_1$ that do not obey our WKB-ansatz for v_x far from the phase velocity ω/k. The effect on $\tilde{E}$ from these terms are negligible, however. See the Appendix for details.

[19] This gives a minimum condition for δv_x that we will come back to later.

[20] This means that once the perturbations are present, the kinetic energy in the resonant region is lower than for the unperturbed state.

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A mechanical analog of first- and second-order phase transitions

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A mechanical model that exhibits first- and second-order phase transitions is analyzed. The possible configurations are found first by using Newtonian mechanics and second by determining the minimum of the effective potential energy taken from the Lagrangian. A comparison is made between the effective potential energy method and the Landau theory of phase transitions. Phase diagrams are obtained for the mechanical system and are compared with those of a ferromagnet.

I. INTRODUCTION

Mechanical models are very useful in explaining more abstract physical concepts. For example, several papers have appeared in this journal in which a mechanical system is used to explain spontaneously broken symmetry[1] and phase transitions.[2–9] In this paper a previous model[1,5] that was used to demonstrate spontaneously broken symmetry and second-order phase transitions is extended to include first-order phase transitions.

We find a close analogy between the mechanical system introduced here and a ferromagnetic material in an external magnetic field. The potential energy of the mechanical system resembles the free energy of the ferromagnet. Also, the phase diagrams are quite similar. So, by considering the possible configurations of this mechanical system we can gain insight into the phases and phase transitions of a magnetic system.

In Sec. II the model is introduced and its solution is described in terms of Newtonian mechanics. There it is shown that a continuous or a discontinuous change in the position can take place, depending on the values of the parameters. In Sec. III the Lagrangian mechanical solution to the model is discussed. Rather than solving the equations of motion, we analyze the effective potential energy to find the possible equilibrium states of the system. In addition, the analogy between the possible states of the present model and phase transitions described by Landau theory is made. Section IV contains a comparison of the phase diagrams of the mechanical model and those of a ferromagnet. Finally, Sec. V includes some conclusions and a discussion.

II. THE MODEL AND ITS ANALYSIS USING NEWTONIAN MECHANICS

The mechanical model to be analyzed[1,5] is shown in Fig. 1. Mass m is free to move without friction on the loop of radius R. The loop is attached at the top to a support that is free to rotate about a vertical axis. In the general case the loop is attached so that the axis of rotation is parallel to the vertical diameter of the loop and is offset a distance A. The position of the mass on the loop is given by the angle θ. The forces acting on the mass are $m\mathbf{g}$, the force of gravity, and $\mathbf{N}$, the normal force due to the loop. When the support, and therefore the loop, is rotating, the mass has centripetal acceleration due to the horizontal component of the normal force.

In order to avoid ambiguity in the definitions of positive and negative θ and A, we define one end of the support as positive and the other end as negative (see Fig. 1). Then the displacement A is positive if the loop shifts toward the positive end of the support. Likewise, θ is positive if the mass swings toward the positive end of the support.

Figure 2 shows the forces applied to mass m. When the loop support is rotating, the net vertical force must be zero, since there is no vertical acceleration. The net force in the

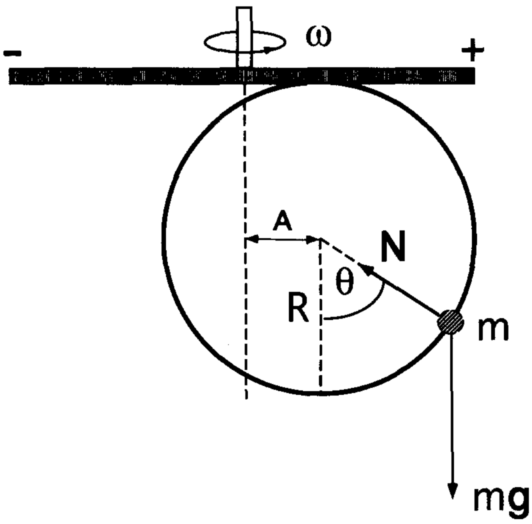

Fig. 1. A schematic of the mechanical model. The end marked (+) defines the positive direction for both A and θ.

horizontal direction produces the centripetal acceleration. Then the Newtonian equations of motion, $\mathbf{F}_{\text{tot}}=m\mathbf{a}$, become

$$\text{vertical component: } N\cos\theta-mg=ma_y=0, \tag{1}$$

$$\text{horizontal component: } N\sin\theta=ma_x=m\omega^2(R\sin\theta+A). \tag{2}$$

By eliminating N in (1) and (2), the resulting equation relating θ and A is found to be

$$mg\sin\theta=m\omega^2(R\sin\theta+A)\cos\theta. \tag{3}$$

Defining $\alpha=A/R$ and $\beta=\omega^2R/g$, we get

$$\sin\theta=\beta(\sin\theta+\alpha)\cos\theta. \tag{4}$$

When (4) is satisfied we say, for convenience, that the system is "in equilibrium." This is not strictly true, since the mass has centripetal acceleration toward the axis of rotation.[10] However, in a reference frame rotating with the loop the object will be stationary. So, we will use the term "equilibrium angle" to describe the possible positions of the mass.

We now look at solutions of (4) for different values of the parameters α and β.

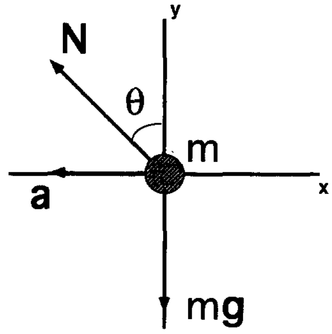

Fig. 2. A force diagram for the mass on the rotating loop: **a** is the centripetal acceleration.

A. $\alpha=0$, β changes

We first consider the case, as discussed in Refs. 1 and 5, when $\alpha=0$, or when the axis of rotation coincides with the diameter of the loop. In this case (4) becomes

$$\sin\theta=\beta\sin\theta\cos\theta. \tag{5}$$

Then the equilibrium angle is

$$\theta_0=\pm\cos^{-1}(1/\beta)=\pm\cos^{-1}(g/\omega^2R). \tag{6}$$

If $g/\omega^2R>1$, or $\beta<1$, then (6) has no real solution for θ_0. However, (5) also allows the trivial solution $\sin\theta=0$ or $\theta=0$. So, for $\beta\geqslant1$ or $\omega\geqslant(g/R)^{1/2}$, the equilibrium angle is given by (6) and for $\beta<1$, or $\omega<(g/R)^{1/2}$, the equilibrium angle is $\theta_0=0$.

As β increases from zero, the mass will remain at $\theta=0$. This is the state with lowest energy and the system is in stable equilibrium. When β exceeds 1, the system is no longer in the state with lowest energy. Since a small force will cause the mass to move into the state with lowest energy, the system is in unstable equilibrium. The new stable equilibrium position is at θ_0 given by (6). As β increases further the mass will continue to move up the loop, with θ varying continuously with β. Note that the mass can move initially in either the positive or negative θ direction.

B. Fixed $\alpha\neq0$, β changes

If $\beta=0$, then $\theta=0$ and the mass m starts out at rest at a distance $A=\alpha R$ from the axis of rotation. Taking α to be positive, if we increase β the equilibrium angle will increase continuously from 0 to a positive value. Physically we expect the mass to swing out away from the axis of rotation for nonzero α when the system is rotating. This can also be seen from an analysis of the equilibrium condition in (4), which can be rewritten as

$$\beta=\frac{\tan\theta}{\alpha+\sin\theta}. \tag{7}$$

We can determine how θ changes with β by differentiation of (7). Then

$$\frac{d\beta}{d\theta}=\frac{(\alpha+\sin\theta)\sec^2\theta-\tan\theta\cos\theta}{(\alpha+\sin\theta)^2} \tag{8}$$

or

$$\frac{d\theta}{d\beta}=\frac{(\alpha+\sin\theta)^2}{\alpha+\sin^3\theta}\cos^2\theta. \tag{9}$$

If $\beta=0$, then $\theta=0$ for any α. The mass hangs straight down if the system is not rotating. As β increases, since for $\theta=0$, $d\theta/d\beta=\alpha$, then θ increases initially in the direction of α. From (9) it is clear that if α and θ are in the same direction, then $d\theta/d\beta>0$. So, as the system rotates faster, the angle θ increases.

C. Fixed $\beta\neq0$, α changes

Suppose $\beta\neq0$ and $\alpha=0$. What happens when α increases in the positive direction? We can see how θ changes with α by solving (4) for α:

$$\alpha=\frac{1}{\beta}\tan\theta-\sin\theta. \tag{10}$$

Differentiation with respect to θ gives

$$\frac{d\alpha}{d\theta}=\frac{1}{\beta}\sec^2\theta-\cos\theta \tag{11}$$

or

$$\frac{d\theta}{d\alpha}=\frac{\cos^2\theta}{1/\beta-\cos^3\theta}. \tag{12}$$

Clearly $d\theta/d\alpha$ can be positive or negative. We now consider two possibilities: $\beta<1$ and $\beta>1$.

1. $\beta<1$

In this case, when $\alpha=0$, $\theta=0$. (See Sec. II A) Since $\beta<1$, $1/\beta>1$, and since $\cos^3\theta\leqslant 1$, we have $1/\beta-\cos^3\theta>0$. Then, from (12) we have $d\theta/d\alpha>0$. So, for $\alpha>0$, θ increases in the positive direction and for $\alpha<0$, θ increases in the negative direction. This is true for all $0\leqslant\theta\leqslant\pi/2$, so θ will continue to increase as α increases. This is essentially the same as the situation in Sec. II B. As α increases, the mass swings out further.

2. $\beta>1$

In this case it is possible for the denominator in (12) to be zero. This will happen when

$$\theta=\theta_c=\pm\cos^{-1}(1/\beta^{1/3}). \tag{13}$$

This gives the following possibilities:

$$\frac{d\theta}{d\alpha}=\begin{cases}<0 & \text{if } |\theta|<\theta_c\\ >0 & \text{if } |\theta|>\theta_c\end{cases}. \tag{14}$$

With $\beta>1$ and $\alpha=0$, suppose θ starts out at $+\theta_0=+\cos^{-1}(1/\beta)>\theta_c$. Then, as α increases in the *positive* direction, $d\theta/d\alpha>0$ and θ increases in the positive direction.

If θ starts out at $+\theta_0$ and α becomes *negative*, then θ decreases and eventually reaches θ_c, where $d\theta/d\alpha$ goes to infinity. Physically this can be described as follows. As α becomes negative, the axis of rotation moves toward the mass and the centripetal acceleration of the mass decreases. A point is reached where the component of the force of gravity exceeds the amount necessary to produce, through the normal force of the loop, the centripetal acceleration. The mass moves toward the axis of rotation, thus decreasing the centripetal acceleration, which causes the mass to "fall" further toward the axis of rotation, eventually moving to the negative side of the loop. Now the mass is "on the other side" of the axis of rotation and it will continue to move until equilibrium is reached at a negative θ. This is a discontinuous change from $+\theta_c$ to a negative θ between $-\theta_0$ and $-\pi/2$.

Another way of describing the above situation is as follows. The range of values of θ from $+\theta_c$ to $+\theta_0$ correspond to states of metastable equilibrium, since there are other stable states at negative values of θ with lower energy. For $\theta=\theta_c$ the mass is still on the positive side of the axis of rotation. As α becomes more negative, the mass must jump to a negative θ between $-\theta_0$ and $-\pi/2$, corresponding to a position of stable equilibrium.

The value α_c at which the discontinuous change in θ takes place can be found as follows. Equation (10) relates α to θ and it can be used to find α_c in terms of θ_c. Then,

$$\alpha_c=\frac{1}{\beta}\tan\theta_c-\sin\theta_c=\left(\frac{1}{\beta\cos\theta_c}-1\right)(1-\cos^2\theta_c)^{1/2}. \tag{15}$$

Now, from (13), $\theta_c=\cos^{-1}(\beta^{-1/3})$, so

$$\alpha_c=-(1-\beta^{-2/3})^{3/2}. \tag{16}$$

The preceding analysis also applies to the case where θ starts out at $-\theta_0$. Then, if α increases in the *negative* direction, θ will increase in the negative direction. If α becomes *positive*, there will be a discontinuous change from $-\theta_c$ to a positive θ between $+\theta_0$ and $+\pi/2$ at $\alpha_c=+(1-\beta^{-2/3})^{3/2}$.

III. THE LAGRANGIAN, THE EFFECTIVE POTENTIAL ENERGY, AND PHASE TRANSITIONS

In this section we set up the Lagrangian for the model shown in Fig. 1. The potential energy is then used to describe the possible equilibrium states of the system and a comparison of this description with the Landau theory of phase transitions is made.

A. The Lagrangian

The Lagrangian is defined as $\mathscr{L}=T-V$, where T is the total kinetic energy and V is the total potential energy.[10] The total kinetic energy is the sum of the kinetic energies of motion around the axis of rotation and along the loop, since the two velocities are perpendicular. Then,

$$T=T_\theta+T_\omega=\tfrac{1}{2}mR^2\dot{\theta}^2+\tfrac{1}{2}m\omega^2(R\sin\theta+A)^2. \tag{17}$$

The only potential energy is gravitational, so choosing the center of the loop as the zero,

$$V=-mgR\cos\theta, \tag{18}$$

and the Lagrangian is

$$\mathscr{L}=\tfrac{1}{2}mR^2\dot{\theta}^2+\tfrac{1}{2}m\omega^2(R\sin\theta+A)^2+mgR\cos\theta. \tag{19}$$

Equation (19) can be written as $\mathscr{L}=T_\theta-U$, where U is an effective potential energy that includes the effects of gravity and of the rotation of the system. Using the definitions introduced previously, $\alpha=A/R$ and $\beta=\omega^2R/g$, we define the *scaled* effective potential energy as

$$U_{\text{eff}}=U/mgR=-\cos\theta-\tfrac{1}{2}\beta(\sin\theta+\alpha)^2. \tag{20}$$

The Lagrangian can then be used to obtain the equations of motion. However, since we are only interested in the equilibrium states, we can analyze U_{eff} for possible solutions. Equilibrium states correspond to states of minimum potential energy. To find these states we will consider the different possible cases for different values of the parameters α and β. First, however, we briefly describe the Landau theory of phase transitions.

B. Phase transitions and Landau theory

The Landau theory[11–13] is fairly successful in describing systems near a second-order phase transition. In this theory the phase of a system is characterized by an order parameter. This is a measurable quantity which is typically zero in the disordered, or high-temperature phase and is nonzero in the ordered, or low-temperature phase. A common example is the ferromagnet. The magnetization of a ferromagnet at a temperature above the critical temperature T_c is zero if there is no external magnetic field present. However, as the temperature is lowered to $T<T_c$, the magnetization becomes nonzero even in the absence of an external magnetic field. Thus the magnetization can be used as an order parameter to distinguish the high- and low-temperature phases in a ferromagnet.

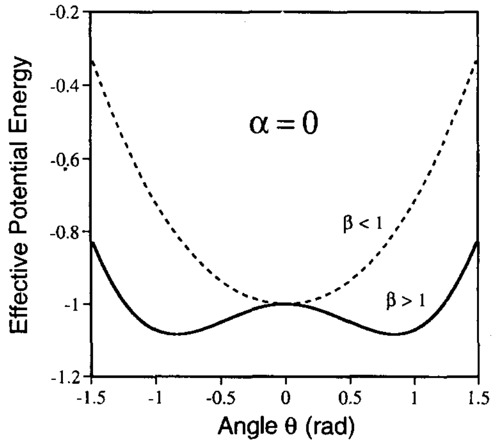

Fig. 3. Effective potential energy curves for loop rotating about its vertical diameter. Curves are for angular frequency below and above the critical value.

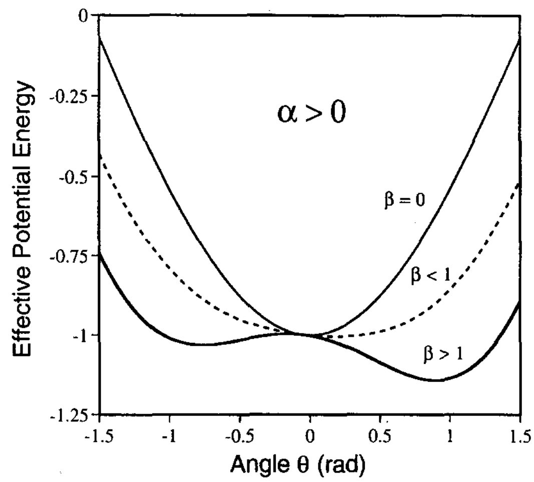

Fig. 4. Effective potential energy curves for loop rotating about an axis parallel to its vertical diameter. Curves for three different angular frequencies are plotted.

When one of the parameters in a system changes, a phase transition may take place. For example, changing the temperature of a ferromagnet can cause the magnetization to become nonzero. The magnetization can change smoothly or abruptly and the phase transition is classified as either second order (continuous) or first order (discontinuous), respectively. Thus, in a second-order phase transition, the order parameter changes continuously, while in a first-order transition, the order parameter changes discontinuously.

To describe phase transitions, Landau considered an expansion of the free energy $\mathcal{F}$ of a system in the order parameter ϕ. The general form of the expansion can be written[11–13]

$$\mathcal{F}=a_1\phi+a_2\phi^2+a_3\phi^3+a_4\phi^4+\cdots, \tag{21}$$

where a constant a_0 has been dropped. We have assumed that the order parameter is constant so that terms including gradients of ϕ have been neglected. The a_i's will, in general, be functions of temperature T. The equilibrium state is then determined by the requirement that $\partial\mathcal{F}/\partial\phi=0$. To determine the stability of the equilibrium state, the second derivative of $\mathcal{F}$ must be evaluated. If $\partial^2\mathcal{F}/\partial\phi^2$ is negative, then the state is unstable. If $\partial^2\mathcal{F}/\partial\phi^2$ is positive, then the state is either stable or metastable, with the stable state having the lower energy. This theory can also be used to describe first-order transitions in the presence of an external field.

The mechanical model considered here can be described by analogy with phase transitions. The position of the mass on the loop, denoted by θ, can be used as an order parameter. Likewise, the angular frequency of rotation ω, or equivalently β, plays the role of temperature. The displacement of the axis of rotation away from the diameter of the circle, represented by a or α, is analogous to the external magnetic field. Finally, the effective potential energy is similar to the free energy. With these identifications, the possible equilibrium configurations of the mass on the loop can be described in terms of phases of the system and the possibility of phase transitions exists. We will henceforth use the term "temperature" to refer to either β or T and the term "field" for either α or the magnetic field B. "Energy" will be used for U_{eff} or $\mathcal{F}$ and "order parameter" will mean either θ or the magnetization M.

The effective potential energy given by (20) can be expanded about $\theta=0$ for small θ and small α. Then,

$$U_{\text{eff}}\simeq-1-\tfrac{1}{2}\beta\alpha^2-\alpha\beta\theta+\frac{1}{2}(1-\beta)\theta^2+\frac{\alpha\beta}{6}\theta^3+\frac{1}{6}(\beta-1/4)\theta^4. \tag{22}$$

This is similar to the Landau expansion with odd and even powers of the order parameter θ and with coefficients which depend on the frequency β.

The possible states of the mechanical system can be determined by examining the energy and how it changes with temperature and external field. A metastable or stable equilibrium state corresponds to a local energy minimum, while a stable equilibrium state corresponds also to a global energy minimum. In the next three subsections we consider the possible equilibrium states for different combinations of the temperature and the field.

C. Zero field, changing temperature

In this case the energy is given by

$$U_{\text{eff}}=-\cos\theta-\tfrac{1}{2}\beta\sin^2\theta. \tag{23}$$

This energy is plotted in Fig. 3 for $\beta<1$ and $\beta>1$.

For the situation in which $\beta<1$ the minimum of the energy is at $\theta_0=0$. This is the equilibrium position of mass m. This will be true for any $\beta<1$, which corresponds to $\omega<(g/R)^{1/2}$.

When $\beta>1$, Fig. 3 shows that the minimum of the energy is no longer at $\theta=0$. There are now minima at

$$\theta_0=\pm\cos^{-1}(1/\beta)=\pm\cos^{-1}(g/\omega^2R) \tag{24}$$

and $\theta=0$ is a local maximum. Therefore, the equilibrium position of mass m when $\beta>1$ will be *either* $+\theta_0$ or $-\theta_0$.

This case was examined in Sec. II A. It is analogous to a second-order phase transition, since as β changes from <1 to >1, the order parameter changes continuously from $\theta=0$ to $\theta\neq0$. In the Landau theory for a ferromagnet this corresponds to an expansion in even powers of M:

$$\mathcal{F}=a_2M^2+a_4M^4, \tag{25}$$

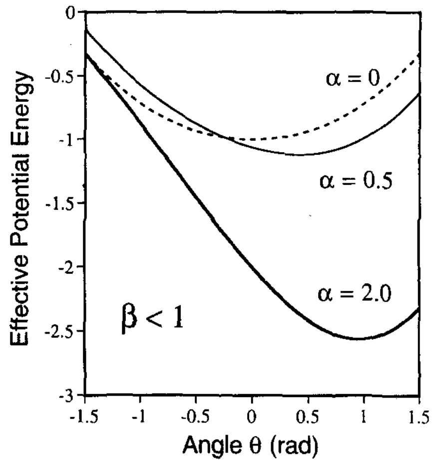

Fig. 5. Effective potential energy curves for loop rotating at a fixed angular frequency below the critical value. Curves for three different displacements of the loop are plotted.

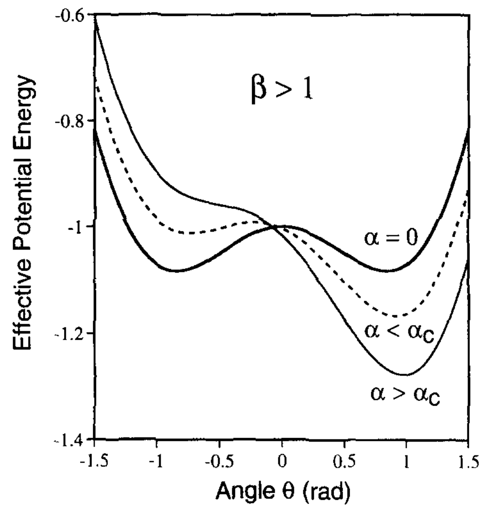

Fig. 6. Effective potential energy curves for loop rotating at a fixed angular frequency above the critical value. Curves for three different displacements of the loop are plotted.

where $a_2=b(T-T_c)$ with b a constant. When $T>T_c$ a plot of $\mathcal{F}$ vs M looks like the $\beta<1$ curve in Fig. 3. When $T<T_c$, the plot resembles the $\beta>1$ curve in Fig. 3. The magnetic material has changed from nonmagnetic to magnetic, with M either positive or negative. The small-angle expansion in (22) with $\alpha=0$ is

$$U_{\text{eff}}=-1+\tfrac{1}{2}(1-\beta)\theta^2+\tfrac{1}{6}(\beta-1/4)\theta^4. \tag{26}$$

This has the same form as (25).

D. Nonzero field, changing temperature

In this case the effective potential energy is given by

$$U_{\text{eff}}=-\cos\theta-\tfrac{1}{2}\beta(\sin\theta+\alpha)^2. \tag{27}$$

This is plotted in Fig. 4 for $\alpha>0$ and $\beta=0$, <1, and >1. Note that the $\beta>1$ curve has two minima. However, since for low β the minimum is at positive θ, this will continue to be true as β increases. The mass cannot be at $\theta<0$ in this case. So, for fixed $\alpha>0$ and nonzero β, the mass will be at an angle $\theta>0$ and as β increases the order parameter θ will increase continuously.

The magnetic analog of this case corresponds to the presence of a fixed external magnetic field and a changing temperature. As T decreases, the magnetization increases in the direction of the external field and, as T passes through T_c, there is no phase transition.

A plot of the free energy versus magnetization will resemble Fig. 4 for infinite temperature ($\beta=0$), $T>T_c$ ($\beta<1$), and $T<T_c$ ($\beta>1$). The Landau expansion for this case would be (25) with a linear term corresponding to the presence of the magnetic field:

$$\mathcal{F}=b(T-T_c)M^2+a_4M^4-BM. \tag{28}$$

For the mechanical model, the small angle expansion is

$$U_{\text{eff}}=-(1+\beta\alpha^2/2)-\alpha\beta\theta+\tfrac{1}{2}(1-\beta)\theta^2+\frac{1}{6}\,\alpha\beta\theta^3$$
$$+\tfrac{1}{6}(\beta-1/4)\theta^4. \tag{29}$$

This contains a linear term proportional to the displacement α, similar to the magnetic case. U_{eff} also contains a cubic term, which would arise from nonlinear effects in the magnetic case. Therefore, (28) has the same form as (29).

E. Nonzero temperature, changing field

We now consider fixed temperature and we let the field change from zero to a positive value. To find the new equilibrium positions of the mechanical system we analyze (27) for the two cases, $\beta<1$ and $\beta>1$. We then compare the results with the magnetic system.

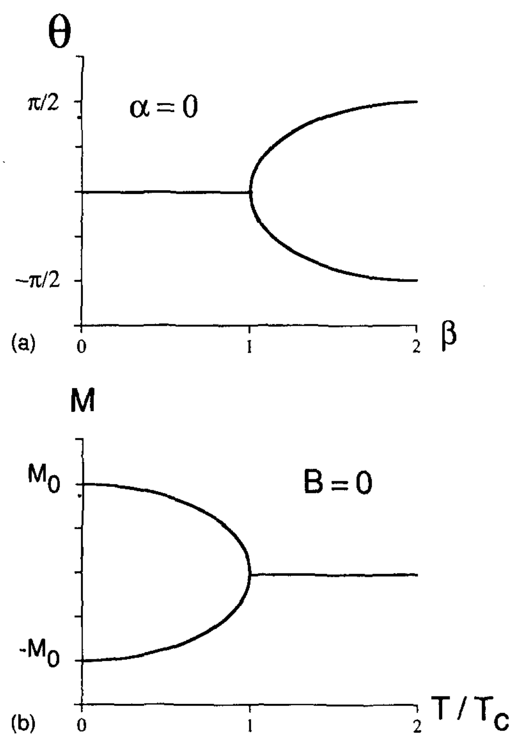

Fig. 7. (a) Position–frequency phase diagram for mechanical system. (b) Magnetization–temperature phase diagram for a ferromagnet.

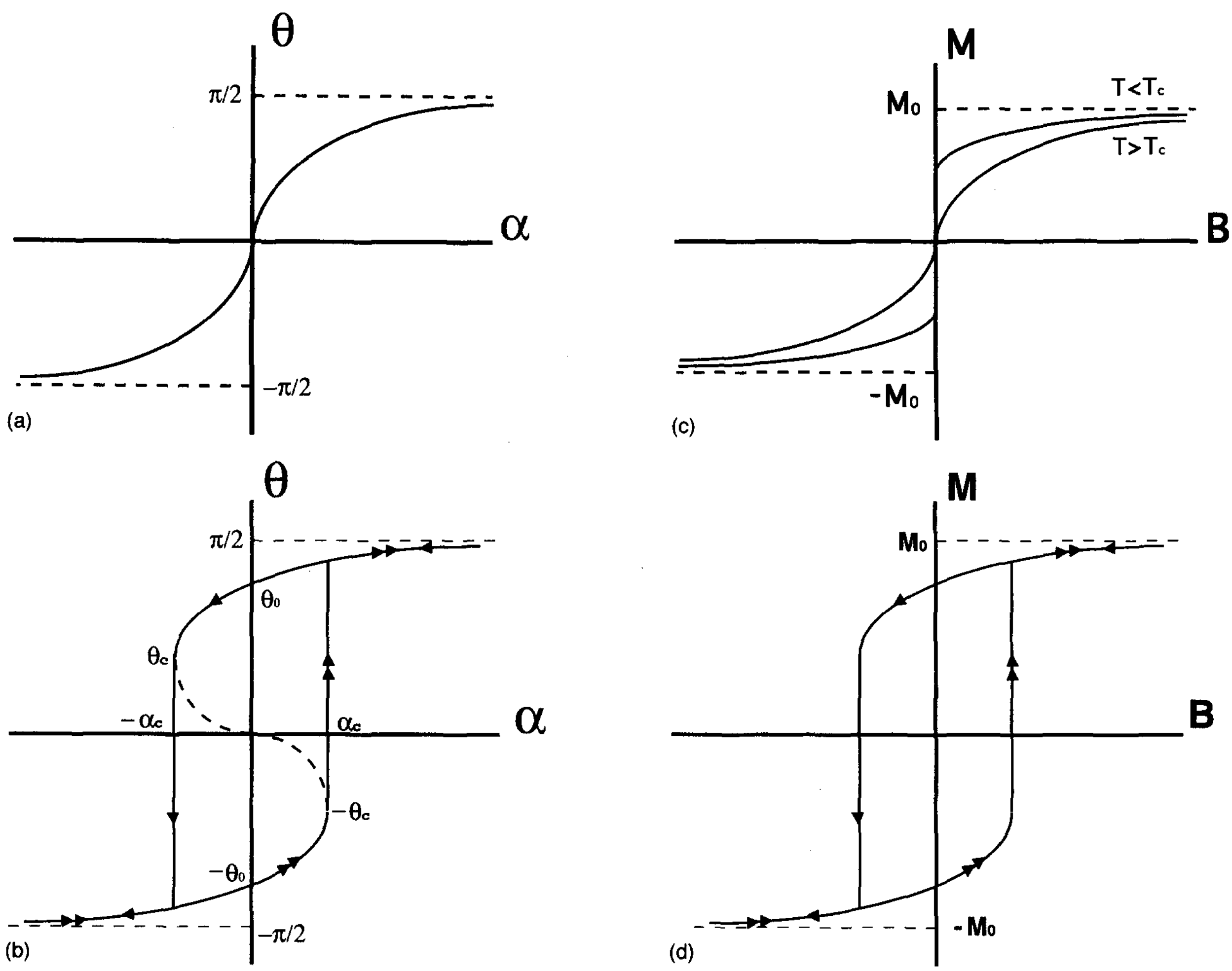

Fig. 8. (a) Position–displacement phase diagram for mechanical system with $\beta<1$. (b) Position–displacement phase diagram for mechanical system with $\beta>1$. (c) Magnetization-field phase diagrams for a ferromagnet with $T<T_c$ and $T>T_c$ and without hysteresis. (d) Magnetization-field phase diagram for a ferromagnet with $T<T_c$ and with hysteresis.

1. β<1

This case is plotted in Fig. 5 for $\beta<1$ and $\alpha=0$, 0.5, and 2. As α increases, the minimum shifts to larger positive values. Therefore, as the loop moves further away from the axis of rotation in the positive direction, the mass moves further up the loop on the side where $\theta>0$.

For the magnetic system this corresponds to $T>T_c$ and B changing from 0 to a positive value. The magnetization starts at zero and increases with B and is parallel to B. The energy varies with magnetization as U_{eff} varies with θ in Fig. 5.

2. β>1

This case is plotted in Fig. 6 for $\beta>1$ and for three different values of α. The curve with $\alpha=0$ is just the solution with minima at $\pm\theta_0$. Suppose the mass chooses $-\theta_0$. Then, as α becomes positive the minima shift and the minimum at positive θ has lower effective potential energy than the negative θ minimum. Thus the minimum at negative θ corresponds to metastable equilibrium. The mass will remain at negative θ until α reaches a critical value, α_c, at which point the minimum at negative θ becomes a point of inflection and the mass slides to the minimum at positive θ.

For the analogous magnetic system the temperature is below T_c and the magnetization is nonzero and negative. If a magnetic field is now applied *in the direction of* M, the magnetization will increase. If a magnetic field is applied *in the direction opposite* M, the magnetization will discontinuously change from negative to positive, assuming no hysteresis (see Sec. IV B). This is an example of a first-order phase transition. The free energy versus magnetization graph will resemble that in Fig. 6 with $\alpha\neq0$.

IV. PHASE DIAGRAMS

In this section we continue the comparison of the mechanical system and magnetic system by looking at their phase diagrams. In particular we look at order parameter–temperature, order parameter–field, and field–temperature diagrams.

A. Order parameter–temperature diagrams

Figure 7(a) shows the θ–β phase diagram for the mechanical system with no displacement of the loop ($\alpha=0$). Then, for $\beta<1$, the mass is at $\theta=0$. As β passes through $\beta=1$, the mass begins to move up the loop. Note that it can go in either the positive or negative θ direction. As β increases further, θ approaches the maximum values of $\pm\pi/2$.

To find how θ varies with β near $\beta=1$, we observe [see (24)] that

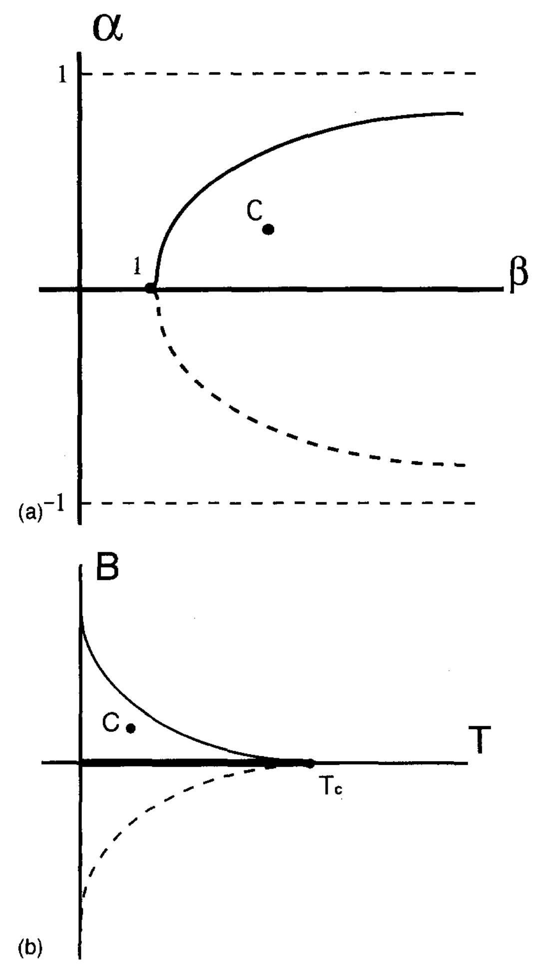

Fig. 9. (a) Displacement–frequency phase diagram for mechanical system. (b) Field–temperature phase diagrams for an ideal and for a real ferromagnet.

$$\cos(\theta_0)=1/\beta. \tag{30}$$

For β near to but greater than 1, θ_0 will be small and we can expand $\cos\theta_0$ in a Taylor series. Then,

$$1-\frac{\theta_0^2}{2}\simeq\frac{1}{\beta}$$

and

$$\theta_0\simeq 2^{1/2}(1-\beta^{-1})^{1/2}. \tag{31}$$

Figure 7(b) shows the magnetization–temperature phase diagram for a ferromagnet with no external field. When $T>T_c$, the magnetization is zero. When T decreases below T_c, the magnetization increases continuously from zero. Note that it can be positive or negative. As T decreases further, M approaches the saturation values $\pm M_0$. Landau theory[12] predicts that as T approaches T_c *from below*, M goes to zero as $(T_c-T)^{1/2}$. This has the same form as (31). However, for real magnets the exponent is closer to $\frac{1}{3}$. This shows one of the limitations of the Landau theory.

B. Order parameter–field phase diagrams

Figure 8(a) shows θ vs α for the mechanical system for $\beta<1$. θ changes continuously from positive to negative values as α goes from positive to negative. Figure 8(b) shows the case where $\beta>1$. Here θ changes discontinuously from θ_c to $-\pi/2<\theta<-\theta_0$ at $\alpha=-\alpha_c$ (following the path with single arrowheads), or from $-\theta_c$ to $\theta_0<\theta<\pi/2$ at $\alpha=\alpha_c$ (following the path with double arrowheads).

Figure 8(c) shows plots of M vs B for a ferromagnet for $T<T_c$ and $T>T_c$. The curve for $T>T_c$ is of the same form as Fig. 8(a), which corresponds to $\beta<1$ for the mechanical system.

The curve in Fig. 8(c) for $T<T_c$ is different from Fig. 8(b), which corresponds to $\beta>1$ for the mechanical system. The reason for this is that the $T<T_c$ curve in Fig. 8(c) is for an ideal ferromagnet. Real ferromagnets are composed of domains, which are small regions in which the magnetization is a maximum. (See Ref. 14.) For the ideal ferromagnet with $T<T_c$, the magnetization changes direction when the external field changes from a small positive to a small negative value. In real ferromagnets, a finite field is necessary to change the direction of the magnetization. This is shown if Fig. 8(d). Following the path with the single arrowhead, the magnetization remains positive until a negative field is reached, at which point the magnetization jumps to a negative value. If the field is now slowly reversed, the ferromagnet follows the path with the double arrowhead. This figure demonstrates the phenomena of hysteresis in ferromagnets. Figure 8(d) is of the same form as Fig. 8(b), showing the analogy between the mechanical model and a real ferromagnet.

C. Field–temperature phase diagrams

The field–temperature phase diagram of the mechanical system is shown in Fig. 9(a). The solid curve represents a line of first-order phase transitions; above the curve θ is positive and below the curve θ is negative. The solid curve is called a coexistence curve and the free energies of the two phases are equal on it. It terminates on the β axis at $\beta=1$, which is called a critical point. A system passing through this point will undergo a second-order phase transition.[12,13]

Note that Fig. 9(a) is not a true phase diagram due to the fact that the value of the phase, given by θ, cannot be determined from the diagram. For example, the value of θ at point C depends on how the system got there. This is a consequence of the fact that for a given $\beta>1$ and a given $-\alpha_c<\alpha<\alpha_c$, there are two possible values of θ. This is shown in Fig. 8(b) and is reflected in the hysteresis of the system. So, the solid line in Fig. 9(a) is the coexistence curve for the situation when the point C has $\theta<0$. Then moving in the direction of increasing α the system reaches the coexistence curve and undergoes a first-order phase transition to $\theta>0$.

If the point C in Fig. 9(a) corresponds to $\theta>0$, then the dashed curve below the β axis is the coexistence curve and it terminates at the same critical point as the solid curve.

Figure 9(b) is a graph of magnetic field B versus temperature T for a ferromagnet. The solid line along the T axis is the coexistence curve for the ideal ferromagnet. Above this curve the magnetization is positive and below the curve it is negative. This is a line of first-order phase transitions and it terminates at a critical point at $T=T_c$.

For a real ferromagnet the coexistence curve will resemble the solid curve above the T axis. As with the mechanical model, the point C does not have a unique value for the order parameter. This is reflected in the hysteresis shown in Fig. 8(d). The solid curve above the T axis is the coexistence curve when the point C has $M<0$. Then, increasing B with

$T<T_c$ takes the system through the coexistence curve and M changes discontinuously to $M>0$. The dashed curve below the T axis is the coexistence curve for the point C having $M>0$.

V. SUMMARY AND DISCUSSION

We have demonstrated that a mechanical model can simulate both first- and second-order phase transitions for certain values of the parameters. This was shown by direct analysis of the equations of motion and by examination of the minima of the effective potential energy. It was then demonstrated that the latter method is similar to the Landau theory of continuous phase transitions. The energy-position graphs in Figs. 3–6 are of the same general shape as those in Ref. 15 for a ferromagnet and Ref. 16 for a general free energy.

The analogy between the phase diagrams of the mechanical model and the ferromagnet is also very close if the hysteresis properties of the ferromagnet are included. Equivalently, if the mass m could tunnel from the higher energy local minimum to the lower in the metastable state, the mechanical system would simulate an ideal ferromagnet. The quantum mechanical version of this model ($\alpha=0$) is considered in Ref. 17.

Finally, it would be interesting to build a working model of this system. Although finding a method for making quantitative measurements may require some imagination, the qualitative aspects of the phase transitions would be easy to observe.

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Magnetic dipole oscillations and radiation damping

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We consider the problem of radiation damping for a magnetic dipole oscillating in a magnetic field. An equation for the radiation reaction torque is derived, and the damping of the oscillations is described. Also discussed are runaway solutions for a rotating magnetic dipole moving under the influence of the reaction torque, with no external torque.

I. INTRODUCTION

When a compass needle is put in the earth's magnetic field, it oscillates about its equilibrium position for a few seconds before coming to rest. The energy of oscillation has been dissipated by friction. Even if there were no friction, however, the oscillations would still be damped, although very slowly, because a compass needle is an oscillating magnetic dipole and radiates electromagnetic energy.

Figure 1 shows a magnetic dipole, with magnetic moment $\mathbf{m}$, which is free to rotate in the x,y plane about a pivot fixed at the origin. The magnetic moment $\mathbf{m}(t)$ is

$$\mathbf{m}(t)=m_0(\cos\,\phi(t)\hat{\mathbf{i}}+\sin\,\phi(t)\hat{\mathbf{j}}). \tag{1}$$

The equation of motion is that the torque equals the rate of change of angular momentum, $\mathbf{N}=d\mathbf{L}/dt$. The torque is $\mathbf{N}=\mathbf{m}\times\mathbf{B}=-m_0B\sin\phi\,\hat{\mathbf{k}}$ for the magnetic field $\mathbf{B}=B\hat{\mathbf{i}}$; the angular momentum is $\mathbf{L}=I\dot{\phi}\hat{\mathbf{k}}$, where I is the moment of

