Hula-Hoop: An Example of Heteroparametric Excitation

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This paper considers the parametric excitation of a pendulum swinging in a horizontal plane. It is shown that there exist a number of different limit cycle motions, one of which is a steady rotation about the point of support. This motion is associated with the mechanism whereby a Hula-Hoop may be kept in rotation by an oscillatory motion of the point of support. The stability and dependence of this type of motion on the initial conditions are analyzed in detail.

INTRODUCTION

In the past year or two a number of interesting toys such as the Hula-Hoop and Eskimo Yo-Yo have appeared on the American market. The acceptance of these toys has been so widespread that it is unnecessary to describe them to an American reader. This paper is concerned with the basic mechanism by which these toys are made to rotate by applying an oscillatory motion to them. Reduced to their simplest terms, these toys are essentially pendulums which are caused to rotate by an oscillatory motion applied to their point of support.

THEORY

Consider a mathematical pendulum of length l and mass m whose point of support A is caused to move in a prescribed manner along a fixed axis zz', the support A being such that the pendulum can rotate in a horizontal plane.

Let θ be the angle which the pendulum makes with the fixed axis zz' (see Fig. 1).

EQUATION OF MOTION

Applying Lagrange's equations to the system shown in Fig. 1, the equation of motion is

$$ml^2\ddot{\theta}+\beta\dot{\theta}-ml\ddot{x}\sin\theta=0, \quad (1)$$

where $\beta\dot{\theta}$ represents the damping which is present in all physical systems.

If in (1), the acceleration $\ddot{x}$ is constant, the equation describes the classical pendulum. In the case of the Hula-Hoop, the acceleration $\ddot{x}$ is a periodic function of time. Let

$$x=x_0\cos\omega t, \quad (2)$$

so that (1) becomes

$$ml^2\ddot{\theta}+\beta\dot{\theta}+m\omega^2 lx_0\cos\omega t\sin\theta=0. \quad (3)$$

Dividing through (3) by ml^2 and letting

$$\beta/ml^2=\zeta, \quad (4)$$

Eq. (3) becomes

$$\ddot{\theta}+\zeta\dot{\theta}+\omega^2 x_0/l\sin\theta\cos\omega t=0. \quad (5)$$

Since it contains time explicitly, Eq. (5) becomes a system with heteroparametric excitation.

POSSIBLE MOTIONS

Analysis of Eq. (5) reveals that three possible motions may exist.

(a) Static Equilibrium $\theta=0, \pi$

As will be shown in the Appendix, this solution is valid provided $x_0/l<0.5$.

(b) Steady Oscillation about $\theta=0$ or $\theta=\pi$

This motion will be studied in the Appendix; it will be shown that this kind of motion may exist only in the range $0.5<x_0/l<0.58$.

(c) Steady Rotation about Point of Support

A third kind of motion is a quasi-steady rotation about the point of support. This is the kind of motion which is of special interest in the Hula-Hoop problem, and hence will form the main topic of this paper. Let

$$\theta=\omega t+\phi, \quad (6)$$

where ϕ is a slowly varying function of time. Substitute (6) into (5) to obtain

$$d^2\phi/dt^2+\omega\zeta+\zeta d\phi/dt +\omega^2x_0/2l[\sin\phi+\sin(2\omega t+\phi)]=0. \quad (7)$$

EQUATION OF THE MEAN

Since it was assumed that ϕ was slowly varying, Eq. (7) may be averaged over one cycle of the base acceleration $\ddot{x}$ as follows:

$$(d^2\phi/dt^2)_{Av}+\omega\zeta+\zeta(d\phi/dt)_{Av}+\omega^2x_0/2l\sin\phi_{Av}=0, \quad (8)$$

where ϕ_{Av} is the average value of ϕ over one cycle of $\ddot{x}$.

It should be noted in passing that if $\zeta \ll 1$ and $\omega^2x_0/2l \ll 1$, then, since $\sin\phi_{Av}$ is a bounded function, ϕ_{Av} must be slowly varying.

STEADY-STATE MOTION

The steady-state rotational motion corresponds to $(d^2\phi/dt^2)_{Av}=(d\phi/dt)_{Av}=0$.

Hence (8) becomes

$$\zeta\omega+\omega^2x_0/2l\sin(\phi_0)_{Av}=0, \quad (9)$$

the zero subscript denoting the steady state. From (9),

$$\sin(\phi_0)_{Av}=-2\zeta l/\omega x_0. \quad (10)$$

Real solutions of (10) exist if, and only if,

$$|2\zeta l/\omega x_0|<1.$$

It will be assumed in the analysis that follows that $|2\zeta l/\omega x_0|=\gamma<1$. With this restriction

$$(\phi_0)_{Av}=-\sin^{-1}\gamma+2r\pi, \quad \sin^{-1}\gamma+(2r+1)\pi \quad (r=0, 1, 2, \cdots). \quad (11)$$

STABILITY OF STEADY-STATE MOTION

Let η be a small perturbation and set

$$\phi_{Av}=(\phi_0)_{Av}+\eta. \quad (12)$$

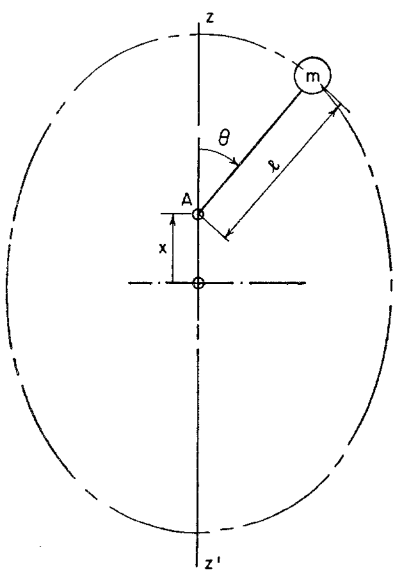

FIG. 1. The physical pendulum, equivalent to the Hula-Hoop.

If we substitute (12) into (8) and use the steady-state equation (9), Eq. (8) becomes

$$\ddot{\eta}+\zeta\dot{\eta}+\omega^2x_0/2l\cos(\phi_0)_{Av}\eta=0. \quad (13)$$

The conditions under which (13) is stable are

(i) $\zeta>0$,

(ii) $\omega^2x_0/2l\cos(\phi_0)_{Av}>0$.

For real physical systems (i) is automatically satisfied. If $(\phi_0)_{Av}=-\sin^{-1}\gamma+2r\pi$, $\cos(\phi_0)_{Av}>0$ and Eq. (13) defines a stable system; if $(\phi_0)_{Av}=(2r+1)\pi+\sin^{-1}\gamma$, $\cos(\phi_0)_{Av}<0$ and Eq. (13) defines an unstable system.

Since $r\geqslant 1$ merely rotates the pendulum through an angle $2\pi r$, it is sufficient to consider only the case for $r=0$. Thus Hula-Hoop type motion is possible if, and only if, the pendulum lags behind the motion of the support by an angle $\sin^{-1}\gamma$ which lies between zero and $\pi/2$.

INFLUENCE OF INITIAL CONDITIONS

To study the influence of initial conditions on the steady-state motion it is necessary to eliminate the explicit dependence on time from Eq. (8); to this end let

$$p=\dot{\phi}_{Av}. \quad (14)$$

Equation (8) may now be written as two first-order equations:

$$\left.\begin{aligned}\dot{p}&=-\omega\zeta-\zeta p-\omega^2x_0/2l\sin\phi_{Av}\\ \dot{\phi}_{Av}&=p\end{aligned}\right\}. \quad (15)$$

Hence

$$dp/d\phi_{Av}=-(\omega\zeta+\zeta p+\omega^2x_0/2l\sin\phi_{Av})/p. \quad (16)$$

SINGULAR POINTS OF EQ. (16)

The singular points of Eq. (16) are determined by the simultaneous vanishing of numerator and denominator. Thus

$$\left.\begin{aligned}\omega\zeta+\zeta p_0+(\omega^2x_0/2l)\sin(\phi_0)_{Av}&=0\\ p_0&=0\end{aligned}\right\}, \quad (17)$$

that is,

$$\left.\begin{aligned}2\zeta l/\omega x_0=\gamma&=-\sin(\phi_0)_{Av}\\ p_0&=0\end{aligned}\right\}. \quad (18)$$

Thus

$$\left.\begin{aligned}&(\phi_0)_{Av}=-\sin^{-1}\gamma+2r\pi, \quad \sin^{-1}\gamma+(2r+1)\pi\\ &p_0=0, \quad (r=0, 1, 2, \cdots)\end{aligned}\right\}. \quad (19)$$

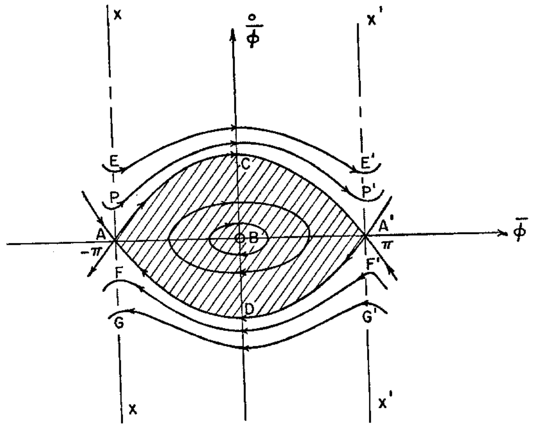

FIG. 2. Integral curves (undamped case). In this and following figures, ϕ with a bar over it is equivalent to ϕ_{Av} as it occurs in the text.

CLASSIFICATION OF SINGULAR POINTS

By using standard techniques[1] the singular points are easily classified. They are as follows:

Point	Nature of singularity
$p_0=0,\quad (\phi_0)_{Av}=-\sin^{-1}\gamma+2r\pi$	Focal point—stable
$p_0=0,\quad (\phi_0)_{Av}=\sin^{-1}\gamma+(2r+1)\pi$	Saddle point—unstable

In general Eq. (16) must be integrated by graphical or numerical techniques. If $\zeta=0$, however, Eq. (16) is an exact differential and may be integrated straightforth; thus

$$\zeta=0,\quad \tfrac{1}{2}p^2-(\omega^2x_0/2l)\cos\phi_{Av}=\text{const.} \qquad (20)$$

In the case in which $\zeta=0$, the singular points are $(\phi_0)_{Av}=2r\pi$, $(2r+1)\pi$, and the focal points become vortex points.

EQUATION OF SEPARATRIX

The equation of the separatrix is obtained by passing the integral curve through any of the saddle points:

$$\left.\begin{matrix}(\phi_0)_{Av}=(2r+1)\pi\\ p_0=0\end{matrix}\right\}\quad (r=0,1,2,\cdots).$$

Thus the constant in Eq. (20) $=\omega^2x_0/2l$.

Since ϕ_{Av} and p are periodic with period 2π, it is convenient to use cylindrical phase space[1] to plot the integrals of motion. This has been done in Fig. 2, the separatrix is indicated by $ACA'D$. Stable motion about the vortex point B is achieved if the initial conditions lie within the shaded area bounded by the separatrix.

[1] N. Minorsky, *Introduction to Non-Linear Mechanics* (Edwards Brothers, Ann Arbor, Michigan, 1947).

INTEGRALS OF MOTION FOR DAMPED CASE

In the case where the damping is not zero the integrals of motion must be obtained by graphical or numerical integration. This has been done and the results are shown qualitatively in Fig. 3. In this case, stable motions converging to the focal point B are obtained for initial conditions lying within the shaded regions. It is interesting to note in passing that if ϕ is initially greater than zero, i.e., if $|\dot{\theta}|>\omega$, there is a good chance of "capture" with stable Hula-Hoop motion resulting. If however, ϕ is initially less than zero, i.e., if $|\dot{\theta}|<\omega$, there is very little possibility of capture.

EXPERIMENTAL RESULTS

To verify the results of the above theory, the rotating pendulum experiment was set up in the Dynamics laboratory at Caltech; the base motion being achieved by means of an M.B. electromagnetic shake table. The pendulum was illuminated by means of a General Radio Strobotac so that the phase relationship between the pendulum and the support could be studied. It was found that:

(a) With small damping, the pendulum rotated in phase with the motion of the support.

(b) If the initial angular velocity of the pendulum was higher than ω, the angular frequency of the excitation, the pendulum was "captured" in approximately half the tests. If the initial angular velocity was lower than ω, the pendulum was never "captured."

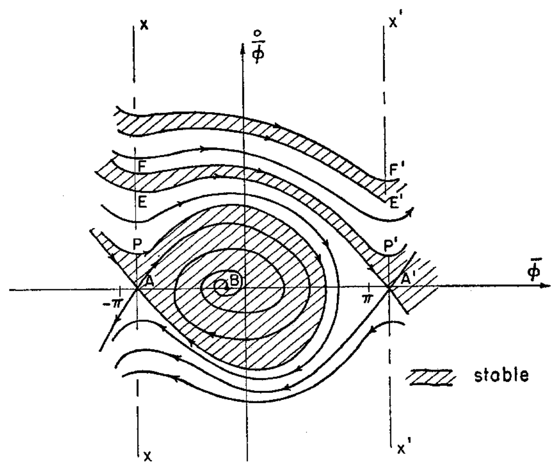

FIG. 3. Integral curves (damped case).

These results fully substantiate the theory given in the foregoing.

CONCLUSIONS

The foregoing analysis provides a partial explanation of the mechanism whereby toys such as the Hula-Hoop and the Eskimo Yo-Yo are made to rotate by oscillating their point of support. The actual mechanism is somewhat more complicated by virtue of the fact that the person holding the toy is at liberty to change the frequency of the oscillating support at will and thereby accommodate a wider range of initial conditions.

APPENDIX

Possible Motions

As was stated in the main body of this paper, three possible motions may exist:

(*a*) *static equilibrium* $\theta=0,\ \pi$;
(*b*) *steady oscillation about* $\theta=0$, *or* $\theta=\pi$;
(*c*) *steady rotation about point of support.*

Case a. Stability

Let $\zeta=0$ for simplicity, then it will be noted that $\theta_0=0,\ \pi$ are solutions of Eq. (5).

Let $\theta=\theta_0+\eta$, $\theta_0=0,\ \pi$, η small. Equation (5) becomes

$$\ddot{\eta}+(\omega^2x_0/l)\cos\theta_0\cos\omega t\ \eta=0. \tag{22}$$

Let $2z=\omega t$, then

$$d^2\eta/dz^2+[4(x_0/l)\cos\theta_0\cos 2z]\eta=0. \tag{23}$$

Equation (23) is of the form

$$d^2\eta/dz^2+(a+2q\cos 2z)\eta=0, \tag{24}$$

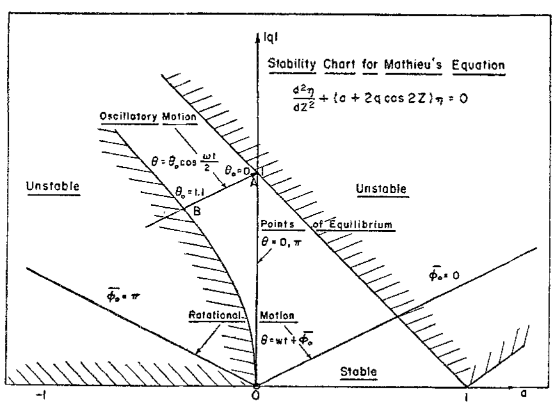

FIG. 4. Stability chart for Mathieu's equation.

which is Mathieu's equation.[2] In this case $a=0$, $|q|=2x_0/l$. If first-order theory is used to compute the unstable regions of the Mathieu equation,[3] it may be shown that Eq. (23) is stable if

$$|q|<1,\quad \text{i.e., if}\quad x_0/l<0.5. \tag{25}$$

This is illustrated in Fig. 4. Hence the equilibrium positions $\theta=0,\ \pi$ are stable provided $x_0/l<0.5$.

Case b. Steady Oscillation about Point of Support

If in (*a*) $x_0/l>0.5$, the point (a,q) lies in the first unstable region of the Matheiu equation. In this region η has the form $\eta=e^{\lambda z}\cos(z+\phi)$. This suggests that a finite amplitude solution of the form $\theta=\theta\cdot\cos[(\omega t/2)+\phi]$ be used in Eq. (5). Now,

$$\sin\theta=\sin\left[\theta_0\cos\left(\frac{\omega t}{2}+\phi\right)\right]$$

$$=2J_1(\theta_0)\cos\left(\frac{\omega t}{2}+\phi\right)$$

$$-2J_3(\theta_0)\cos\left(\frac{3\omega t}{2}+3\phi\right)-\cdots. \tag{26}$$

If $\theta_0<\pi/2$,

$$\sin\theta\simeq 2J_1(\theta_0)\cos\left(\frac{\omega t}{2}+\phi\right) \tag{27}$$

(J_n = Bessel function of the first kind, of order n).

Hence Eq. (5) with $\zeta=0$ becomes

$$-\frac{\omega^2}{4}\theta_0\cos\left(\frac{\omega t}{2}+\phi\right)+\frac{\omega^2x_0}{l}J_1(\theta_0)$$

$$\times\left[\cos\left(\frac{\omega t}{2}-\phi\right)+\cos\left(\frac{3\omega t}{2}+\phi\right)\right]=0. \tag{28}$$

Equating coefficients of $\cos[(\omega t/2)+\phi]$ and $\sin[(\omega t/2)+\phi]$ separately to zero, we get

$$-\frac{\omega^2}{4}\theta_0+\frac{\omega^2x_0}{l}J_1(\theta_0)\cos 2\phi=0, \tag{29}$$

$$\frac{\omega^2x_0}{l}J_1(\theta_0)\sin 2\phi=0. \tag{30}$$

[2] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Oxford University Press, London, 1947).
[3] C. Hayashi, *Forced Oscillations in Non-Linear Systems* (Nippon Printing Company, Osaka, Japan, 1953).

Solutions to (29) exist for $0<\theta_0<\pi$ only if $2\phi=0,\ 2\pi,\ 4\pi$. Hence (29) becomes

$$\theta_0=(4x_0/l)J_1(\theta_0). \tag{32}$$

Solutions exists only for $(x_0/l)>0.5$.

Stability of Oscillation Motion

By letting η be a small perturbation on θ, we may set

$$\theta=\theta_0\cos\left(\frac{\omega t}{2}+\phi\right)+\eta. \tag{33}$$

Equation (5) with $\zeta=0$ then becomes

$$\ddot{\eta}+x_0\omega^2/l\left[\cos\left\{\theta_0\cos\left(\frac{\omega t}{2}+\phi\right)\right\}\cos\omega t\right]\eta=0. \tag{34}$$

Because

$$\cos\left\{\theta_0\cos\left(\frac{\omega t}{2}+\phi\right)\right\}$$
$$=J_0(\theta_0)-2J_2(\theta_0)\cos(\omega t+2\phi)\cdots, \tag{35}$$

Equation (34) becomes

$$\ddot{\eta}+x_0\omega^2/l[-J_2(\theta_0)\cos2\phi+J_0(\theta_0)\cos\omega t$$
$$-J_2(\theta_0)\cos(2\omega t+2\phi)-\cdots]\eta=0. \tag{36}$$

Let $2z=\omega t$, then

$$\frac{d^2\eta}{dz^2}+4x_0/l[-J_2(\theta_0)\cos2\phi+J_0(\theta_0)\cos2z$$
$$-J_2(\theta_0)\cos(4z+2\phi)]\eta=0. \tag{37}$$

Since $2\phi=0,\ 2\pi,\ 4\pi$, Eq. (37) becomes

$$\frac{d^2\eta}{dz^2}+4x_0/l[-J_2(\theta_0)+J_0(\theta_0)\cos2z$$
$$-J_2(\theta_0)\cos4z]\eta=0. \tag{38}$$

Equation (38) is of the form

$$\frac{d^2\eta}{dz^2}+[a+\sum_{\nu=1}^{\infty}2q_\nu\cos2\nu z]\eta=0, \tag{39}$$

which is a Mathieu-Hill equation.

If the discussion is restricted to values of $\theta_0<\pi/2$, (38) may be approximated by a Mathieu equation with

$$\left.\begin{aligned}a&=-(4x_0/l)J_2(\theta_0)\\ q&=(2x_0/l)J_0(\theta_0)\end{aligned}\right\}. \tag{40}$$

By using Eq. (32),

$$4x_0/l=\theta_0/J_1(\theta_0), \tag{41}$$

and combining (41) and (40),

$$\left.\begin{aligned}a&=-\theta_0J_2(\theta_0)/J_1(\theta_0)\\ q&=\tfrac{1}{2}\theta_0J_0(\theta_0)/J_1(\theta_0)\end{aligned}\right\}. \tag{42}$$

By using (42) it is possible to calculate a and q parametrically and to plot them in the stability chart to determine when the motion is stable. This has been done in Fig. 4. From curve AB it will be seen that if $\theta_0>1.1$ radians the system is unstable. The corresponding value of x_0/l at point B is 0.58. Thus the oscillatory motion is stable only in the range $0.5<x_0/l<0.58$.

Case c. Steady Rotation about Point of Support

In addition to motions (a) and (b), it was shown in the first part of this paper that rotation about the point of support was also possible. It was shown that if $\zeta=0$,

$$(\theta_0)_{Av}=\omega t+(\phi_0)_{Av},\ (\phi_0)_{Av}=0,\pi. \tag{43}$$

Stability of Rotational Motions

Let $\theta=(\theta_0)_{Av}+\eta$, where η is a small perturbation. Equation (5) becomes

$$\ddot{\eta}+\omega^2x_0/2l\{\cos(\phi_0)_{Av}+\cos[2\omega t+(\phi_0)_{Av}]\}\eta=0. \tag{44}$$

Letting $2\omega t+(\phi_0)_{Av}=2z$,

$$(d^2\eta/dz^2)+x_0/2l[\cos(\phi_0)_{Av}+\cos2z]\eta=0. \tag{45}$$

This is a Mathieu equation with

$$\left.\begin{aligned}a&=(x_0/2l)\cos(\phi_0)_{Av}\\ |q|&=(x_0/4l)\end{aligned}\right\}, \tag{46}$$

hence

$$a=2|q|\cos(\phi_0)_{Av}. \tag{47}$$

Equation (47) is plotted in the stability chart of Fig. 4. It will be seen that $(\phi_0)_{Av}=0$ is stable if $a=x_0/2l<0.67$, i.e., if $x_0/l<1.34$. If $(\phi_0)_{Av}=\pi$, the system is unstable for all values of x_0/l.

It should be noted in passing that the Mathieu equation approach verifies the analysis based on the equations of the mean. It is interesting to note in passing that if in Sec. (b) $x_0/l>0.58$, the point (a,q) lies in an unstable region where η takes the form $\eta=e^{\lambda z}$. This suggests that a finite amplitude solution of the form $\theta=\omega t+\phi$ be used in Eq. (5). The analysis of Sec. (*c*) shows that such a motion would be stable.

Experimental Results

To verify the results of (*a*) and (*b*) above, the experimental setup used in the first part of the paper was again used. It was found that:

(i) If the pendulum was initially at rest and the support was oscillated, the pendulum would take up an equilibrium position corresponding to $\theta=0$ or π, provided the amplitude of oscillation of the support was small enough.

(ii) If the amplitude of oscillation of the support was increased, a point was reached at which the pendulum began to oscillate about the equilibrium position. Further, it was observed, by means of the stroboscope, that the frequency of oscillation was exactly half that of the support.

(iii) If the amplitude of oscillation of the support was increased still further, a point was reached at which the oscillatory motion became unstable and the pendulum began to rotate about the point of support in Hula-Hoop fashion. The values of x_0/l corresponding to these different types of motion were found to be in qualitative agreement with the theory.

Extending the Lorentz Transformation by Characteristic Coordinates

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The problem considered is that of rectilinear motion with variable velocity. The paper gives, by an elementary construction, a system of coordinates which is conformal in a restricted region near the axis of the motion. In such coordinates the velocity of light remains invariant even for observers moving with variable velocity. By a particular choice of the scale relation the restricted conformal transformations can be made to reduce to the Lorentz transformation everywhere in the case of constant velocity and locally in the case of variable velocity.

IN the *American Journal of Physics*, November, 1958, Leffert and Donahue call attention to irregularities that appear when the Lorentz transformation is extended to problems of variable motion. Figure 1 illustrates the difficulty alluded to. Here the moving origin of a system B is plotted as a curvilinear world line on a rectangular system which is not shown, but which we may designate as A. In such a diagram the lines $t'=$constant associated with B are oblique and if they are continued as straight lines they will cross, leading to a nonuniform correspondence of events between the A and B systems. This lack of uniformity appears in the conventional treatments of the problem, as, for example, in the analysis given by Møller.[1]

A uniform correspondence can be achieved, however, if the Lorentz transformation is extended by means of characteristic lines, rather than along straight t' lines. An extension along straight t' lines amounts to the assumption that the Lorentz transformation propagates instantaneously in the B system and at the electromagnetic phase velocity c^2/v in the A system. The characteristic lines, however, have the same slope in either system, and of course propagate at the velocity of light. The use of the characteristic lines establishes a conformal correspondence between the two systems x, it and x', it'. As is well known, such transformations preserve a constant velocity of light during accelerated motions, even in three-dimensional space, if they can be established.[2,3] This note shows how such coordinates can be established in the vicinity of the line of motion for a system with variable rectilinear velocity.

Figure 2 shows the curvilinear coordinates obtained in the xt plane when the Lorentz

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[1] C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1952), pp. 258–263.

[2] H. Bateman, *Electrical and Optical Wave Motion* (Dover Publications, New York, 1955), S14.

[3] L. Infeld and A. Schild, Phys. Rev. **26**, 250–272 (1945).