Physics 106a/196a – Midterm Exam – Due Nov 3, 2006
Solutions

Andrey Rodionov, Peng Wang, Sunil Golwala

Version 2: November 7, 2006

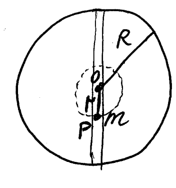
version 2: Correct error in Eqn 22 in solution to Problem 4 (algebra error).

Problem 1

(1

Problem 1 (Particle inside a sphere with spherically symmetric distribution of mass density $\rho(r)$).

A particle inside the sphere is attracted by the part of the sphere that lies within a shell of radius r, where r is particle's distance from the center of the sphere.



Particle with mass m in a point P inside the sphere of radius R with center at point "O" is attracted by the inner sphere of radius r, as if there is no outside spherical layer with radius $(R-r)$ and R

Gravitational force in point P:

(1) $\boxed{F = G\dfrac{M(r)\cdot m}{r^2}}$, direction of F is towards the center "O".

$M(r)$ is mass inside the sphere of radius r.

(2) $\boxed{M(r) = \int_0^r 4\pi (r')^2 \cdot \rho(r')\cdot dr'}$

{More precisely: $M(r) = \int_0^r \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} (r')^2 \sin\theta \cdot d\theta \cdot d\varphi \cdot \rho(r')\, dr' \underset{\text{due to spherical symmetry}}{=} 4\pi \int_0^r r'^2 \rho(r')dr'$

(2

There are several alternative ways how to solve the problem further:

→ For harmonic oscillator:

$\ddot{r} + \omega^2 r = 0$; $\ddot{r} = \dfrac{F}{m}$ because $F = m\ddot{r}$.

$G\dfrac{M(r)}{r^2} + \omega^2 r = 0$

$\Rightarrow \boxed{M(r) \sim r^3}$ OR $\int_0^r 4\pi (r')^2 \cdot \rho(r')\cdot dr' \sim r^3$

$\Rightarrow \boxed{\rho(r) \text{ doesn't depend on } r}$

→ We know that for harmonic oscillations:

$F = -kr$

$\Rightarrow -kr = -G\dfrac{M(r)\cdot m}{r^2}$ (from (1))

$\Rightarrow M(r) = \dfrac{k}{m}\cdot\dfrac{1}{G}\cdot r^3 \sim r^3$

and again we have: $4\pi \int_0^r (r')^2 \rho(r')\cdot dr' = \dfrac{k}{mG}\cdot r^3 = \dfrac{\omega^2}{G} r^3$

Differentiating both parts:

$4\pi r^2 \cdot \rho(r) = 3\dfrac{\omega^2}{G}\cdot r^2$; or $\boxed{\rho(r) = \dfrac{3}{4\pi}\cdot\dfrac{\omega^2}{G}}$

Problem 2

Conservation of energy yields

$$T + Q = T_3 + T_4 \tag{1}$$
$$\Rightarrow T + Q = \frac{p_3^2}{2m_3} + \frac{p_4^2}{2m_4}$$

where T, p are the kinetic energy and the momentum of the incident particle, and $T_{3,4}$, $p_{3,4}$ of the particle $m_{3,4}$. Conservation of momentum along the track of the incident particle is

$$p = p_3 \cos\psi_3 + p_4 \cos\psi_4$$
$$\Rightarrow \sqrt{2m_1 T} = p_3 \cos\psi_3 + p_4 \cos\psi_4 \tag{2}$$

Conservation of momentum perpendicular to the track of the incident particle is

$$p_3 \sin\psi_3 = p_4 \sin\psi_4 \tag{3}$$

Plugging Eq. (3) into Eq. (2) gives us

$$p_3 = \frac{\sqrt{2m_1 T}}{\sin\psi_3 (\cot\psi_3 + \cot\psi_4)} \tag{4}$$
$$p_4 = \frac{\sqrt{2m_1 T}}{\sin\psi_4 (\cot\psi_3 + \cot\psi_4)} \tag{5}$$

which, together with Eq. (1), yields

$$T + Q = \frac{2m_1 T}{(\cot\psi_3 + \cot\psi_4)^2} \left(\frac{1}{2m_3 \sin^2\psi_3} + \frac{1}{2m_4 \sin^2\psi_4} \right) \tag{6}$$
$$\Rightarrow T = Q \frac{(\cot\psi_3 + \cot\psi_4)^2}{m_1 \left(\frac{1}{m_3 \sin^2\psi_3} + \frac{1}{m_4 \sin^2\psi_4} \right) - (\cot\psi_3 + \cot\psi_4)^2}$$

When $Q \to 0$, we note from Eq. (6) that ψ_3 and ψ_4 are not independent and the denominator of Eq. (3) will also go to zero which will make T finite.

Problem 3

(1

Problem 3. (Lagrangian for a relativistic particle)

(1) $L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\,\phi(\vec{r}\,)$;

Let's write Lagrangian (1) in cartesian coordinates:

(2) $L = -mc^2\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}} - q\,\phi(x,y,z)$

→ Canonical momenta are given by $\frac{\partial L}{\partial \dot{q}_k}$,

so

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\cdot\frac{1}{\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}} = \frac{m\dot{x}}{\sqrt{1-\frac{v^2}{c^2}}};$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}\cdot\frac{1}{\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}} = \frac{m\dot{y}}{\sqrt{1-\frac{v^2}{c^2}}};$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}\cdot\frac{1}{\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}} = \frac{m\dot{z}}{\sqrt{1-\frac{v^2}{c^2}}},$$

In compact notation:

$$\boxed{p_i = \frac{m\dot{x}_i}{\sqrt{1-\frac{v^2}{c^2}}}} \quad (i = x, y, z)$$

$$\boxed{p_i = \frac{m\dot{x}_i}{\sqrt{1-\frac{v^2}{c^2}}}}$$

(2

→ Euler-Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0;$$

$$\frac{\partial L}{\partial x} = -q\,\frac{\partial \phi(r)}{\partial x};\quad \frac{\partial L}{\partial y} = -q\,\frac{\partial \phi(r)}{\partial y};\quad \frac{\partial L}{\partial z} = -q\,\frac{\partial \phi(r)}{\partial z}$$

Our task is simplified: we are allowed not to calculate explicitly $\frac{d}{dt}$ term

$$\Rightarrow \frac{d}{dt}\left(m\dot{x}_i\,\frac{1}{\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}}\right) + q\cdot\frac{\partial \phi(r)}{\partial x_i} = 0$$

or

$$\boxed{\frac{d}{dt}\left(\frac{m\dot{x}_i}{\sqrt{1-\frac{v^2}{c^2}}}\right) = -q\cdot\frac{\partial \phi(r)}{\partial x_i}}$$

where $x_i = \{x, y, z\}$ for $i = 1, 2, 3$

→ Calculation of Hamiltonian.

$$H = \sum_k \dot{q}_k\,\frac{\partial L}{\partial \dot{q}_k} - L$$

, where we calculated $\frac{\partial L}{\partial \dot{q}_k}$ in the beginning of the problem.

3

$$H = \frac{1}{\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}}} \left(m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 \right) +$$

$$+ mc^2 \sqrt{1 - \frac{v^2}{c^2}} + q \cdot \phi(x, y, z) \, ;$$

$$H = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{mc^2\left(1 - \frac{v^2}{c^2}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} + q \cdot \phi(x, y, z) \, ;$$

$$\boxed{H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + q \cdot \phi(x, y, z)}$$

Total Hamiltonian is NOT equal~~s~~ to the total relativistic energy of the particle. It has the term taking into account the potential energy of the particle in electrostatic field.

→ Hamiltonian is conserved: $\frac{d}{dt} H = 0$

time t is not contained explicitly in Hamiltonian;

also $\frac{\partial L}{\partial t} = 0$, because $\frac{\partial \phi}{\partial t} = 0$ and $\frac{\partial}{\partial t}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 0$

Problem 4

(a) We employ the coordinates system from Example 2.3 in the lecture notes. The system is invariant under translation along the X coordinate, which corresponds to the transformation $(X, d) \to (X + a, d)$. One can prove this by explicit substitution or simply by noting that the Lagrangian is cyclic in X (but not in d!).

The associated conserved momentum is p_X,

$$p_X = (m + M)\,\dot{X} + m\,\dot{d}\cos\alpha \tag{7}$$

If one used the inertial coordinate system from part (b) of the problem, the transformation is the same in concept, translation along the x direction. The specific transformation is translation of *both* the coordinates, $(x_p, y_p, x_b, y_b) \to (x_p + a, y_p, x_b + a, y_b)$. The conserved momentum is $p_{x_p} + p_{x_b}$,

$$p_{x_p} + p_{x_b} = M\dot{x}_p + m\dot{x}_b$$

In both cases, the conserved momentum is just the total linear momentum in the x direction of the "system of particle" consisting of the plane and the block (Equation 1.16 of the lecture notes). Using Equation 1.17 from the lecture notes, its conservation reflects the fact that there is no *external* force acting along the x direction. This makes sense, as the only external force acting in the problem is gravity acting downward on both the plane and the block. There are of course *internal* forces between the block and plane acting, which cause $\dot{X}$ and $\dot{d}$ (or $\dot{x}_p$ and $\dot{x}_b$) to individually change, but always subject to conservation of horizontal linear momentum.

(b) The kinetic energy is

$$T = \frac{1}{2}M\left(\dot{x}_p^2 + \dot{y}_p^2\right) + \frac{1}{2}m\left(\dot{x}_b^2 + \dot{y}_b^2\right)$$

The potential energy is

$$U = mgy_b + Mgy_p$$

Note that you must include the potential energy in y_p because it is now being considered a dynamical coordinate! This was a frequently made mistake in the exams.

The constraint equations are (taking the (x_p, y_p) to indicate the position of the bottom right corner of the plane, where the angle α is)

$$G_p \equiv y_p = 0$$

$$\frac{(y_b - y_p)}{(x_b - x_p)} = -\tan\alpha \Rightarrow G_b \equiv (y_b - y_p)\cos\alpha + (x_b - x_p)\sin\alpha = 0$$

Note that the rewriting of the second constraint as a sum of terms linear in the coordinates is quite convenient because it makes taking the partial derivatives $\frac{\partial G_p}{\partial q_k}$ easy. Our six equations are

$$\begin{aligned}
\frac{\partial L}{\partial x_b}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_b}+\lambda_p\frac{\partial G_p}{\partial x_b}+\lambda_b\frac{\partial G_b}{\partial x_b}&=0\\
\frac{\partial L}{\partial y_b}-\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_b}+\lambda_p\frac{\partial G_p}{\partial y_b}+\lambda_b\frac{\partial G_b}{\partial y_b}&=0\\
\frac{\partial L}{\partial x_p}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_p}+\lambda_p\frac{\partial G_p}{\partial x_p}+\lambda_b\frac{\partial G_b}{\partial x_p}&=0\\
\frac{\partial L}{\partial y_p}-\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_p}+\lambda_p\frac{\partial G_p}{\partial y_p}+\lambda_b\frac{\partial G_b}{\partial y_p}&=0\\
G_p&=0\\
G_b&=0
\end{aligned}$$

Written out explicitly, they are

$$-m\ddot{x}_b+\lambda_b\sin\alpha=0 \tag{8}$$

$$-mg-m\ddot{y}_b+\lambda_b\cos\alpha=0 \tag{9}$$

$$-M\ddot{x}_p-\lambda_b\sin\alpha=0 \tag{10}$$

$$-Mg-M\ddot{y}_p-\lambda_b\cos\alpha+\lambda_p=0 \tag{11}$$

$$y_p=0 \tag{12}$$

$$(y_b-y_p)\cos\alpha+(x_b-x_p)\sin\alpha=0 \tag{13}$$

We see immediately

$$y_p=0 \quad\Longrightarrow\quad \ddot{y}_p=0 \tag{14}$$

Using the first constraint, we differentiate the second constraint twice to obtain

$$\ddot{y}_b=-\left(\ddot{x}_b-\ddot{x}_p\right)\tan\alpha \tag{15}$$

Combining the x_b and x_p equations, we obtain

$$m\ddot{x}_b+M\ddot{x}_p=0 \quad\Longrightarrow\quad \ddot{x}_p=-\frac{m}{M}\ddot{x}_b \tag{16}$$

We also have from the x_b equation

$$m\ddot{x}_b=\lambda_b\sin\alpha \tag{17}$$

We eliminate y_b and λ_b from the y_b equation:

$$-mg+m\left(\ddot{x}_b-\ddot{x}_p\right)\tan\alpha+m\ddot{x}_b\cot\alpha=0 \tag{18}$$

Then we eliminate x_p:

$$-mg+m\left(1+\frac{m}{M}\right)\ddot{x}_b\tan\alpha+m\ddot{x}_b\cot\alpha=0 \tag{19}$$

Now solve for $\ddot{x}_b$:

$$\ddot{x}_b=\frac{g}{\cot\alpha+\left(1+\frac{m}{M}\right)\tan\alpha}=\frac{g\cos\alpha\sin\alpha}{\cos^2\alpha+\left(1+\frac{m}{M}\right)\sin^2\alpha}=\frac{g\cos\alpha\sin\alpha}{1+\frac{m}{M}\sin^2\alpha} \tag{20}$$

We of course will not penalize you for not reducing the result to this clean form, but you must have written $\ddot{x}_b$ only in terms of g, m, M, and α (no other accelerations or λ's) to get full credit. We may now obtain the other accelerations and the Lagrange multipliers:

$$\ddot{x}_p = -\frac{m}{M}\ddot{x}_b = -\frac{g\cos\alpha\sin\alpha}{\frac{M}{m} + \sin^2\alpha} \tag{21}$$

This reproduces the acceleration $\ddot{X}$ found in the lecture notes because x_p and X are the same coordinate. For y_b, use the relation between $\ddot{y}_b$, $\ddot{x}_b$ and $\ddot{x}_p$:

$$\ddot{y}_b = -\ddot{x}_b\left(1 + \frac{m}{M}\right)\tan\alpha = -\frac{g\left(1 + \frac{m}{M}\right)\sin^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} \tag{22}$$

The results for $\ddot{x}_b$ and $\ddot{y}_b$ match the result found for $\ddot{\vec{r}}_b$ found in the lecture notes. Finally, we may solve for the Lagrange multipliers:

$$\lambda_b = \frac{m\ddot{x}_b}{\sin\alpha} = g\frac{m\cos\alpha}{1 + \frac{m}{M}\sin^2\alpha} \tag{23}$$

$$\lambda_p = Mg + \lambda_b\cos\alpha = g\frac{M\left(1 + \frac{m}{M}\sin^2\alpha\right) + m\cos^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} = g\frac{M + m}{1 + \frac{m}{M}\sin^2\alpha} \tag{24}$$

Constraint forces are

$$\begin{aligned}
N_{x_b} &= \lambda_b\frac{\partial G_b}{\partial x_b} + \lambda_p\frac{\partial G_p}{\partial x_b} \\
&= \frac{mg\cos\alpha\sin\alpha}{1 + \frac{m}{M}\sin^2\alpha} \\
N_{y_b} &= \lambda_b\frac{\partial G_b}{\partial y_b} + \lambda_p\frac{\partial G_p}{\partial y_b} \\
&= \frac{mg\cos^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} \\
N_{x_p} &= \lambda_b\frac{\partial G_b}{\partial x_p} + \lambda_p\frac{\partial G_p}{\partial x_p} \\
&= -\frac{mg\cos\alpha\sin\alpha}{1 + \frac{m}{M}\sin^2\alpha} = -N_{x_b} \\
N_{y_p} &= \lambda_b\frac{\partial G_b}{\partial y_p} + \lambda_p\frac{\partial G_p}{\partial y_p} \\
&= -\frac{mg\cos^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} + g\frac{M + m}{1 + \frac{m}{M}\sin^2\alpha} = g\frac{M + m\sin^2\alpha}{1 + \frac{m}{M}\sin^2\alpha} = Mg
\end{aligned}$$

N_{x_b} and N_{y_b} are the x and y components of what you usually think of as the "normal force" exerted by the plane on the block. N_{x_p} is just the reaction force of the block on the plane in the x direction. N_{y_p} is the force exerted by the flat surface to counter gravity to keep the plane at $y_p = 0$.

One might think that N_{y_p} should include a force to counter the force exerted downward on the plane by the block (the reaction force to N_{y_b}). That is not true because N_{y_p} only has to counter the forces that are present in the potential term in the Lagrangian. But one can actually see this extra needed force *inside* the constraint force. Note that N_{y_p} is the sum of two pieces. The λ_b term is the reaction force of the block on the plane, acting downward, and the λ_p term is the force that the flat surface must exert to counter both gravity and the λ_b force – see the earlier equation that gives $\lambda_p = Mg + \lambda_b \cos\alpha$. We only see Mg in the end because the total constraint force N_{y_p} is the force that must be exerted to counteract the forces *explicit in the Lagrangian.* But, clearly, via the Lagrange multipliers, we can identify the internal forces (which are not written in the Lagrangian) that cancel each other out.

Problem 5

Problem 5. (Generalized Mechanics) 11

$L(q_i, \dot{q}_i, \ddot{q}_i, t)$ — Lagrangian contains derivative terms of order greater than one.

Consider the action $S \equiv \int L(q_i, \dot{q}_i, \ddot{q}_i, t)\, dt$;

$$\delta S \equiv S[q+\delta q] - S[q] = \int_{t_0}^{t_1} L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i, \ddot{q}_i+\delta\ddot{q}_i, t)\,dt - \int_{t_0}^{t_1} L(q_i, \dot{q}_i, \ddot{q}_i, t)\,dt;$$

Let's expand in Taylor series:

$$L(q_i+\delta q_i, \dot{q}_i+\delta\dot{q}_i, \ddot{q}_i+\delta\ddot{q}_i, t) = L(q_i, \dot{q}_i, \ddot{q}_i, t) + \frac{\partial L(q_i, \dot{q}_i, \ddot{q}_i, t)}{\partial q_i}\delta q_i + \frac{\partial L(q_i, \dot{q}_i, \ddot{q}_i, t)}{\partial \dot{q}_i}\delta\dot{q}_i + \frac{\partial L(q_i, \dot{q}_i, \ddot{q}_i, t)}{\partial \ddot{q}_i}\delta\ddot{q}_i + \ldots O(q^2)$$

Substituting this expansion into the expression for δS; $L(q, \dot{q}, \ddot{q}, t)$ cancelles out;

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L(q_i, \dot{q}_i, \ddot{q}_i, t)}{\partial q_i}\delta q_i + \frac{\partial L(q_i, \dot{q}_i, \ddot{q}_i, t)}{\partial \dot{q}_i}\delta\dot{q}_i + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}_i \right) dt$$

2

Let's use the fact that

$$\delta \dot{q}_i = \frac{d}{dt}\delta q_i \;; \quad \delta \ddot{q}_i = \frac{d^2}{dt^2}\delta q_i \;, \quad \text{then}$$

$$\delta S = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i + \frac{\partial L}{\partial \ddot{q}_i}\frac{d^2}{dt^2}\delta q_i\right]dt =$$

$$= \int_{t_0}^{t_1} dt\left(\frac{\partial L}{\partial q_i}\delta q_i\right) + \int_{t_0}^{t_1} dt\cdot\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\delta q_i + \int_{t_0}^{t_1} dt\frac{\partial L}{\partial \ddot{q}_i}\frac{d^2}{dt^2}\delta q_i$$

Using the integration by parts $\Rightarrow$

$$\int_{t_0}^{t_1} dt\cdot\frac{\partial L}{\partial \dot{q}}\cdot\frac{d}{dt}\delta q_i = \cancel{\frac{\partial L}{\partial \dot{q}}\delta q_i\Big|_{t_0}^{t_1}} - \int_{t_0}^{t_1} dt\cdot\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\cdot\delta q_i;$$

0 because $\delta q_i = 0$ in the ends of the line (t_0, t_1)

And
$$\int_{t_0}^{t_1} dt\cdot\frac{\partial L}{\partial \ddot{q}_i}\frac{d^2}{dt^2}\delta q_i = \frac{\partial L}{\partial \ddot{q}_i}\cdot\frac{d}{dt}\delta q_i\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} dt\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{d}{dt}\delta q_i =$$

$$= \cancel{\frac{\partial L}{\partial \ddot{q}}\cdot\delta \dot{q}\Big|_{t_0}^{t_1}}_{0} - \int_{t_0}^{t_1} dt\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{d}{dt}\delta q_i = \ldots = \int_{t_0}^{t_1} dt\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i}\delta q_i;$$

3

Therefore, δS becomes:

$$\delta S = \int_{t_0}^{t_1} dt\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\right)\delta q_i = 0$$

And we arrive at the analogy of Euler-Lagrange equations:

$$\boxed{\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0}$$

→ Let's apply this result to $L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$:

$$\frac{d^2}{dt^2}\left(-\frac{m}{2}q\right) - \frac{d}{dt}(0) - \frac{m}{2}\ddot{q} - kq = 0;$$

$$\Rightarrow \boxed{-m\ddot{q} = kq} \quad \text{or} \quad \ddot{q} = -\frac{k}{m}q$$

→ This system is described by the same equation as harmonic oscillator.

Problem 6

(a) We start with the action for L'

$$S' = \int_{t_0}^{t_1} L' dt = \int_{t_0}^{t_1} \left(L + \frac{d}{dt} F\left(\{q_k\}, t\right) \right) dt$$
$$= \int_{t_0}^{t_1} L dt + F\left(\{q_k\left(t_1\right)\}, t_1\right) - F\left(\{q_k\left(t_0\right)\}, t_0\right)$$

Since $q_k\left(t_1\right)$ and $q_k\left(t_0\right)$ are fixed, so we have

$$\delta S' = \delta S$$

which means L' yields the same Euler-Lagrange equations as L.

(b) The Lagrangian of a particle moving in the electromagnetic field is

$$L = \frac{1}{2} m v^2 - q\left[\phi\left(\overrightarrow{x}\right) - \overrightarrow{A}\left(\overrightarrow{x}\right) \cdot \overrightarrow{v}\right]$$

The gauge transformation of the scalar and vector potential gives us

$$L' = \frac{1}{2} m v^2 - q\left[\phi'\left(\overrightarrow{x}\right) - \overrightarrow{A}'\left(\overrightarrow{x}\right) \cdot \overrightarrow{v}\right]$$
$$= L + q\left[\frac{\partial \psi}{\partial t} + \overrightarrow{\nabla}\psi \cdot \frac{\partial \overrightarrow{x}}{\partial t}\right]$$
$$= L + q\frac{d\psi}{dt}$$

which is one special case of Part (a) with $F = q\psi$. From (a), we know the Euler-Lagrange don't change. You can also see this by noticing that

$$\overrightarrow{E}'\left(\overrightarrow{x}\right) = -\overrightarrow{\nabla}\phi'\left(\overrightarrow{x}\right) - \frac{\partial \overrightarrow{A}'\left(\overrightarrow{x}\right)}{\partial t} = -\overrightarrow{\nabla}\phi\left(\overrightarrow{x}\right) - \frac{\partial \overrightarrow{A}\left(\overrightarrow{x}\right)}{\partial t} = \overrightarrow{E}\left(\overrightarrow{x}\right)$$
$$\overrightarrow{B}'\left(\overrightarrow{x}\right) = \overrightarrow{\nabla} \times \overrightarrow{A}'\left(\overrightarrow{x}\right) = \overrightarrow{\nabla} \times \overrightarrow{A}\left(\overrightarrow{x}\right) + \overrightarrow{\nabla} \times \overrightarrow{\nabla}\psi = \overrightarrow{B}\left(\overrightarrow{x}\right)$$

The result for the change in the Lagrangian is consistent with the result for the effect on the motion according to Part (a) with $F = q\psi$.