Physics 106b/196b – Midterm Exam – Due Feb 9, 2007
Solutions

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v. 2: A number of students have indicated how Problem 3 could be read differently than intended. We add a solution that corresponds to that reading.

Problem 1

The three torsion pendulums are subject to the same torques because the string that is being twisted does not change. Euler's equations give us

$$\tau = I_i \omega_i \quad i = 1, 2, 3$$

where I_1, I_2, and I_3 are the moments of inertial for the cube that is hung from a corner, one from midway along an edge, and one from the middle of a face relative to the wires, respectively. So

$$\omega_i = \frac{\tau}{I_i} \quad i = 1, 2, 3$$

and the ratios of the periods of the three pendulums are

$$T_1 : T_2 : T_3 = I_1 : I_2 : I_3$$

The moment of inertial tensor relative to the center of mass for case (b) where we assume z-axis is along the wire and x-axis and y-axis point through the middles of a face is

$$\mathcal{I}_2 = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

where $I_{xx} = I_{yy} = I_{zz} = I_2$ due to symmetry. Because $\mathcal{I}_2$ is proportional to the identity, it will be invariant under any rotation with the center of mass fixed. Since the axis of the rotation passes through the center of mass for all three cases, we have

$$I_1 = I_2 = I_3$$

So

$$T_1 : T_2 : T_3 = 1 : 1 : 1$$

Problem 2

Just as in the lecture notes, our rotating coordinate system is one fixed to the rotating earth at the location of pendulum, with x pointing east, y pointing north, and z normal to the surface. So the angular velocity vector in the rotating system is

$$\vec{\omega}_E = \omega_E \left(\widehat{y}\cos\lambda + \widehat{z}\sin\lambda\right)$$

where ω_E is the Earth's angular velocity. Let us assume the bob of the pendulum is moving a circular path with radius $l\sin\theta$ in xy plane where l is the length of the pendulum. (Note that we do not need to use an underline to indicate a coordinate representation because we express $\vec{\omega}_E$ in terms of the other vectors $\widehat{y}$ and $\widehat{z}$.) So the position vector of the bob is

$$\vec{r}_E = l\sin\theta\left(\cos\omega t\widehat{x} + \sin\omega t\widehat{y}\right)$$

where ω is the angular velocity of the bob moving a circular path. (Again, not the lack of underlines.) The Coriolis force is

$$\begin{aligned}
\underline{\vec{F}_c} &= -2m\underline{\vec{\omega_E}} \times \frac{d\underline{\vec{r_E}}}{dt} \\
&= -2m\omega_E\left(0, \cos\lambda, \sin\lambda\right) \times \omega\, l\, \sin\theta\left(-\sin\omega t, \cos\omega t, 0\right) \\
&= -2m\omega_E\,\omega\, l\sin\theta\left(-\sin\lambda\cos\omega t, -\sin\lambda\sin\omega t, \cos\lambda\sin\omega t\right)
\end{aligned}$$

The z component of F_c is

$$\left(F_c\right)_z = -2m\omega_E\,\omega\, l\, \sin\theta\cos\lambda\sin\omega t$$

The average of $\left(F_c\right)_z$ is

$$\begin{aligned}
\overline{\left(F_c\right)_z} &= \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left(F_c\right)_z dt \\
&= -2m\omega_E\omega l\sin\theta\cos\lambda\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin\omega t dt \\
&= 0
\end{aligned}$$

The ratio of $\left(F_c\right)_z$ to gravity is

$$\frac{\left(F_c\right)_z}{mg} \sim \frac{\omega_E\omega l\sin\theta}{g} \sim \frac{2\pi}{24\times 60\times 60}\frac{1}{9.8} \sim 10^{-5}$$

where we assume $\omega l\sin\theta \sim O\left(1m/s\right)$. So we see that $\left(F_c\right)_z$ is negligible in magnitude. The angular velocity ω of the bob can be decomposed to $\omega_T + \Omega$ where ω_T is the contribution from the tension $\vec{T}$ in the wire attaching to the bob and Ω from the Coriolis force. The tension $\vec{T}$ is

$$\underline{\vec{T}} = T\left(\cos\theta\widehat{z} - \sin\theta\cos\omega t\widehat{x} - \sin\theta\sin\omega t\widehat{y}\right)$$

First, we ignore the Coriolis force to calculate ω_T, which means $\omega \approx \omega_T$.The vertical component of the force acting on the bob is

$$\begin{aligned}
T\cos\theta &= mg \\
T &= \frac{mg}{\cos\theta}
\end{aligned}$$

The component of $\vec{T}$ in xy plane generates the circular move of the bob and determines ω_T by

$$T\sin\theta = m\omega_T^2 l\sin\theta$$
$$\omega_T = \sqrt{\frac{g}{l\cos\theta}}$$

Taking the Coriolis force into account, the component of $\vec{T}+\vec{F_c}$ in xy plane similarly determines $\omega = \omega_T + \Omega$ by

$$\left|T\left(-\sin\theta\cos\omega t\widehat{x} - \sin\theta\sin\omega t\widehat{y}\right) + 2m\omega_E\omega l\sin\theta\left(\sin\lambda\sin\omega t\widehat{y} + \sin\lambda\cos\omega t\widehat{x}\right)\right| = m\omega^2 l\sin\theta$$
$$\sin\theta\left(T - 2m\omega_E\omega l\sin\lambda\right) \approx \sin\theta\left(T - 2m\omega_E\omega_T l\sin\lambda\right) = ml\left(\omega_T+\Omega\right)^2\sin\theta \approx ml\left(\omega_T^2 + 2\omega_T\Omega\right)\sin\theta$$
$$\frac{mg}{\cos\theta} - 2m\omega_E\omega_T l\sin\lambda \approx ml\left(\frac{g}{l\cos\theta} + 2\omega_T\Omega\right)$$
$$\Omega \approx -\omega_E\sin\lambda$$

where we discard terms of $O(\Omega^2)$ in the second line.

Problem 3

The four-wavevector in the emitter reference frame F is

$$k^\mu = \left(\omega, \vec{k}\right) = 2\pi\nu\,(1, \cos\theta, 0, \sin\theta) = 2\pi\nu \left(1, \frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$$

assuming the light signal travels in xz plane. The Lorentz transformation matrix from F to $\widetilde{F}$ is

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta = \frac{4}{5}$, $\gamma = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3}$, and $\beta\gamma = \frac{4}{3}$. So the four-wavevector in the reference frame $\widetilde{F}$ is

$$\begin{aligned}
\widetilde{k}^\mu &= \Lambda^\mu_{\ \nu} k^\nu \\
&= \left(\gamma k^0 + \beta\,\gamma k^1, \gamma k^1 + \beta\gamma k^0, k^2, k^3\right) \\
&= 2\,\pi\,\nu\,(\gamma + \beta\,\gamma\cos\theta, \gamma\cos\theta + \beta\gamma, 0, \sin\theta) \\
&= 2\,\pi\,\nu \left(\frac{5}{3} + \frac{4}{3}\frac{1}{2}, \frac{5}{3}\frac{1}{2} + \frac{4}{3}, 0, \frac{\sqrt{3}}{2}\right) \\
&= 2\,\pi\,\nu \left(\frac{7}{3}, \frac{13}{6}, 0, \frac{\sqrt{3}}{2}\right) \\
&= 2\,\pi \left(\frac{7}{3}\nu\right) \left(1, \frac{13}{14}, 0, \frac{3\sqrt{3}}{14}\right)
\end{aligned}$$

where you can see that $\widetilde{k}^\mu \widetilde{k}_\mu = 0$ as a check for our algebra. So the light signal propagates with the frequency $\widetilde{\nu} = \nu\gamma\,(\beta\cos\theta + 1) = \frac{7}{3}\,\nu$ in a direction that makes an angle $\widetilde{\theta}$ with the $\widetilde{x}$ axis of

$$\tan\widetilde{\theta} = \frac{\sin\theta}{\gamma\,(\cos\theta + \beta)} = \frac{3\sqrt{3}}{13}$$

One could have read the problem in such a way as to think that the emitter was moving in the $-x$-direction toward the origin when emitting the above light. The problem was, admittedly, confusingly phrased. We will accept such a solution if it is correct. The solution for that sign of β is as follows. The form of the Lorentz transformation matrix is unchanged, it is simply the sign of β that changes. So

$$\begin{aligned}
\widetilde{k}^\mu &= \Lambda^\mu_{\ \nu} k^\nu \\
&= \left(\gamma k^0 + \beta\,\gamma k^1, \gamma k^1 + \beta\gamma k^0, k^2, k^3\right) \\
&= 2\,\pi\,\nu\,(\gamma + \beta\,\gamma\cos\theta, \gamma\cos\theta + \beta\gamma, 0, \sin\theta) \\
&= 2\,\pi\,\nu \left(\frac{5}{3} - \frac{4}{3}\frac{1}{2}, \frac{5}{3}\frac{1}{2} - \frac{4}{3}, 0, \frac{\sqrt{3}}{2}\right) \\
&= 2\,\pi\,\nu \left(1, -\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)
\end{aligned}$$

Remarkably, we find clearly that the frequency is unchanged. The photon now makes an angle with the $+x$-axis of

$$\tan\widetilde{\theta} = \frac{\sin\theta}{\gamma\left(\cos\theta + \beta\right)} = -\sqrt{3}$$

which implies $\tilde{\theta} = 120°$. This somewhat surprising result can be understood in a couple of ways:

- Mathematically, the require $\widetilde{\nu} = \nu$ yields an equation with more than solution. Specifically:

  $$\widetilde{\nu} = \nu \quad \Longrightarrow \quad \gamma\left(1 + \beta\cos\theta\right) = 1$$

  There are two solutions to the above equation. The trivial one is $\beta = 0$, which gives $\gamma = 1$. The other one can be found easily:

  $$\begin{aligned} \frac{1 + \beta\cos\theta}{\sqrt{1 - \beta^2}} &= 1 \\ \left(1 + \beta\cos\theta\right)^2 &= 1 - \beta^2 \\ \beta^2\left(1 + \cos^2\theta\right) + 2\beta\cos\theta &= 0 \\ \beta = -\frac{2\,\cos\theta}{1 + \cos^2\theta} & \qquad \text{or} \quad \beta = 0 \end{aligned}$$

  The first solution yields $\beta = -\frac{4}{5}$ for $\cos\theta = \frac{1}{2}$ as in our case. Now we see that the fact the frequency is unchanged is an accident of the angle θ that was chosen; for some arbitrary θ, one would obtain $\beta \neq \frac{4}{5}$, and thus this interpretation of the geometry of the problem would not have yielded $\widetilde{\nu} = \nu$.

- The intuitive explanation is simply that the doppler shift and the length contraction cancel. Length contraction, which doesn't care about the direction of motion, tells us that the apparent wavelength $\widetilde{\lambda}$ will be smaller than the wavelength in the emitter frame λ. But Doppler shift arises because the emitter is moving between the time that he emits the crests of the wave. In this case, the emitter is moving in the opposite direction as the wave emission. This results in the events corresponding to the emission of peaks being separate in $\widetilde{F}$, thereby stretching the wavelength. The length contraction and the Doppler stretching cancel each other out for this particular choice of θ and β. One can see from the mathematical explanation above that, for any choice of θ, there is one β that yields this effect. Note that β always has the opposite since as $\cos\theta$, meaning that the emitter must always be traveling in a direction opposite to the light (in the emitter's rest frame); this makes sense, as it is the condition that there be Doppler stretching of the wavelength. One could work all this out quantitatively using space-time diagrams, but it must yield the same results as the simple Lorentz transformation executed above.

But the fact that $\widetilde{\nu} = \nu$ for this particular choice of θ and β is not important to the problem; for a different θ one would have obtained some $\widetilde{\nu} \neq \nu$ for $\beta = -\frac{4}{3}$, or for a different (negative) β one would have also obtained $\widetilde{\nu} \neq \nu$ for $\theta = 60°$.

Problem 4

The center of mass of the top remains fixed except for the z motion. So the kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}I_1\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2\\ &+\frac{1}{2}M\dot{z}^2\end{aligned}$$

where the first line is the kinetic energy in the center of mass frame and $\frac{1}{2}M\dot{z}^2$ is the kinetic energy for the center of mass. Unlike the case for the fix top, we have I_1 in T instead of I_{1d} since the kinetic energy is relative to the center of mass in our case while the kinetic energy is relative to the pivot point for the fix top. And, what is more, z can be related to θ by

$$z = l\cos\theta$$

So we have for the kinetic energy

$$T = \frac{1}{2}I_1\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2+\frac{1}{2}Ml^2\sin^2\theta\dot{\theta}^2$$

Just like the fix top, the potential energy is

$$U = Mgl\cos\theta$$

where we discarded the constant term. The Lagrangian is

$$\begin{aligned}L &= T-U\\ &= \frac{1}{2}I_1\dot{\phi}^2\sin^2\theta+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2+\frac{1}{2}\left(Ml^2\sin^2\theta+I_1\right)\dot{\theta}^2-Mgl\cos\theta\end{aligned}$$

ψ and ϕ's canonical momenta, p_ψ and p_ϕ, are same as these in the case for a fix top and are also conserved. So p_ψ and p_ϕ are

$$\begin{aligned}p_\psi &= I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)\\ p_\phi &= I_1\dot{\phi}\sin^2\theta+I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)\cos\theta\end{aligned}$$

The EOM for θ is different and is given by

$$\begin{aligned}&\frac{d}{dt}\left(\left(Ml^2\sin^2\theta+I_1\right)\dot{\theta}\right)\\ &\qquad -\frac{\partial}{\partial\theta}\left(\frac{1}{2}I_1\dot{\phi}^2\sin^2\theta+\frac{1}{2}I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)^2+\frac{1}{2}Ml^2\sin^2\theta\dot{\theta}^2-Mgl\cos\theta\right)=0\\ &\frac{d}{dt}\left(\left(Ml^2\sin^2\theta+I_1\right)\dot{\theta}\right)-\frac{\partial}{\partial\theta}\left(\frac{(p_\phi-p_\psi\cos\theta)^2}{2I_1}+\frac{p_\psi^2}{2I_3}\right)-Ml^2\sin\theta\cos\theta\dot{\theta}^2-Mgl\sin\theta=0\end{aligned}$$

Problem 5

The effective potential is

$$V_{eff}(\theta) = \frac{1}{2I_{1d}} \frac{(p_\phi - p_\psi \cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

where $p_\phi = p_\psi$ for a "sleeping" mode. In order to get the frequency of small oscillations around $\theta = 0$, we need to Taylor expand $V_{eff}(\theta)$ at $\theta = 0$. So we have for a "sleeping" mode

$$\begin{aligned} V_{eff}(\theta) &= \frac{p_\phi^2}{2I_{1d}} \frac{(1-\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \\ &\approx \frac{p_\phi^2}{2I_{1d}} \frac{\left(\frac{\theta^2}{2} + O(\theta^4)\right)^2}{\left(\theta - \frac{\theta^3}{6} + O(\theta^4)\right)^2} + Mgl\left(1 - \frac{\theta^2}{2} + O(\theta^4)\right) \\ &\approx \frac{p_\phi^2}{2I_{1d}} \frac{\frac{\theta^4}{4} + O(\theta^6)}{\theta^2 - \frac{\theta^4}{3} + O(\theta^5)} + Mgl\left(1 - \frac{\theta^2}{2} + O(\theta^4)\right) \\ &\approx \frac{p_\phi^2}{2I_{1d}} \frac{\frac{\theta^2}{4} + O(\theta^4)}{1 - \frac{\theta^2}{3} + O(\theta^3)} + Mgl\left(1 - \frac{\theta^2}{2} + O(\theta^4)\right) \\ &\approx \frac{p_\phi^2}{2I_{1d}} \frac{\theta^2}{4} + Mgl\left(1 - \frac{\theta^2}{2}\right) + O(\theta^4) \\ &\approx \theta^2\left(\frac{p_\phi^2}{8I_{1d}} - \frac{Mgl}{2}\right) + Mgl + O(\theta^4) \end{aligned}$$

The EOM is

$$I_{1d}\ddot{\theta} + \theta\left(\frac{p_\phi^2}{4I_{1d}} - Mgl\right) \approx 0$$

So the frequency is

$$\omega = \sqrt{\frac{\frac{p_\phi^2}{4I_{1d}} - Mgl}{I_{1d}}} = \sqrt{\frac{p_\phi^2}{4I_{1d}^2} - \frac{Mgl}{I_{1d}}}$$

Problem 6

In the instantaneous rest frame of the rocket, the four-momentum for the rocket is

$$m\,(1,0)$$

at $t=0$ assuming the rocket travels along $+x$-axis. After the rocket expels gases of mass Δm at $t=\Delta t$, the four-momentum for the rocket and the exhaust gases are respectively

$$\begin{aligned} \text{Rocket:} &\quad \gamma_{dv}\,(m+dm)\,(1,dv,0,0) \\ \text{Gasses:} &\quad \gamma_a \Delta m\,(1,-a,0,0) \end{aligned}$$

where $\gamma_{dv}=\frac{1}{\sqrt{1-dv^2}}$, $\gamma_a=\frac{1}{\sqrt{1-a^2}}$ and dv is the velocity of the rocket in F at $t=\Delta t$. And we also have

$$\begin{aligned} dm &= -\alpha dt \\ \Delta m &= -dm-\kappa \end{aligned}$$

The conservation of four-momentum gives us

$$m\,(1,0,0,0)=\gamma_{dv}\,(m+dm)\,(1,dv,0,0)+\gamma_a\,(-dm-\kappa)\,(1,-a,0,0)$$

which yields

$$\begin{aligned} m &= \gamma_{dv}\,(m+dm)+\gamma_a\,(-dm-\kappa)\approx(m+dm)+\gamma_a\,(-dm-\kappa) \\ 0 &= \gamma_{dv}\,(m+dm)\,dv-\gamma_a\,(-dm-\kappa)\,a\approx mdv-\gamma_a\,(-dm-\kappa)\,a \end{aligned}$$

where we have dropped the second order terms, dv^2 and $dmdv$. The first line leads to

$$dm=\gamma_a\,(dm+\kappa)$$

Plugging it into the second line, we have

$$mdv=-\gamma_a\,(dm+\kappa)\,a=-adm$$

When the rocket picks up dv in F, the velocity of the rocket in $\widetilde{F}$ becomes

$$\begin{aligned} \widetilde{v}+d\widetilde{v} &= \frac{dv+\widetilde{v}}{1+dv\widetilde{v}} \\ \widetilde{v}+d\widetilde{v} &\approx (dv+\widetilde{v})\,(1-\widetilde{v}dv)\approx\widetilde{v}+\left(1-\widetilde{v}^2\right)dv \\ d\widetilde{v} &= \left(1-\widetilde{v}^2\right)dv \end{aligned}$$

So we get

$$\begin{aligned} \frac{d\widetilde{v}}{1-\widetilde{v}^2} &= -a\frac{dm}{m} \\ m\frac{d\widetilde{v}}{dm}+a\left(1-\widetilde{v}^2\right) &= 0 \end{aligned}$$

Problem 7

The initial conditions for ω_1, ω_2 and ω_3 at $t = 0$ are

$$\begin{aligned}
\omega_1(0) &= \Omega\cos\alpha \\
\omega_2(0) &= 0 \\
\omega_3(0) &= \Omega\sin\alpha
\end{aligned}$$

Euler's equations yield

$$\begin{aligned}
I_1\frac{d}{dt}\omega_1 &= \omega_2\omega_3\left(I_2 - I_3\right) \\
I_2\frac{d}{dt}\omega_2 &= \omega_1\omega_3\left(I_3 - I_1\right) \\
I_3\frac{d}{dt}\omega_3 &= \omega_1\omega_2\left(I_1 - I_2\right)
\end{aligned}$$

Taking $\frac{I_1}{I_2} = \cos 2\alpha$ and $I_3 = I_1 + I_2$ into account, Euler's equations become

$$\frac{d}{dt}\omega_1 = -\omega_2\omega_3 \tag{1}$$

$$\frac{d}{dt}\omega_2 = \omega_1\omega_3 \tag{2}$$

$$\frac{d}{dt}\omega_3 = \omega_1\omega_2\left(\frac{I_1 - I_2}{I_1 + I_2}\right) = \omega_1\omega_2\frac{\cos 2\alpha - 1}{1 + \cos 2\alpha} = -\omega_1\omega_2\tan^2\alpha \tag{3}$$

$\omega_1\times$Eq.(1)+ω_2Eq.(2) gives us

$$\begin{aligned}
\frac{d}{dt}\left(\omega_1^2 + \omega_2^2\right) &= 0 \\
\Rightarrow \omega_1^2\left(t\right) + \omega_2^2\left(t\right) &= \Omega^2\cos^2\alpha \\
\Rightarrow \omega_1^2\left(t\right) &= \Omega^2\cos^2\alpha - \omega_2^2\left(t\right) \\
\Rightarrow \omega_1\left(t\right) &= \sqrt{\Omega^2\cos^2\alpha - \omega_2^2\left(t\right)} \qquad (4)
\end{aligned}$$

$\omega_2\tan^2\alpha\times$Eq.(1)+$\omega_3$Eq.(3) gives us

$$\begin{aligned}
\frac{d}{dt}\left(\omega_2^2\tan^2\alpha + \omega_3^2\right) &= 0 \\
\Rightarrow \omega_2^2\left(t\right)\tan^2\alpha + \omega_3^2(t) &= \Omega^2\sin^2\alpha \\
\Rightarrow \omega_3^2(t) &= \Omega^2\sin^2\alpha - \omega_2^2\left(t\right)\tan^2\alpha \\
\Rightarrow \omega_3(t) &= \tan\alpha\sqrt{\Omega^2\cos^2\alpha - \omega_2^2\left(t\right)} \qquad (5)
\end{aligned}$$

Plugging Eq. (4) and Eq. (5) into Eq. (2), we have

$$\begin{aligned}
\frac{d}{dt}\omega_2 &= \tan\alpha\left(\Omega^2\cos^2\alpha - \omega_2^2\left(t\right)\right)\\
\frac{d\omega_2}{\Omega^2\cos^2\alpha - \omega_2^2\left(t\right)} &= \tan\alpha dt\\
\frac{d\frac{\omega_2}{\Omega\cos\alpha}}{1-\frac{\omega_2^2(t)}{\Omega^2\cos^2\alpha}} &= \Omega\sin\alpha dt\\
\int_0^{\frac{\omega_2(t)}{\Omega\cos\alpha}}\frac{du}{1-u^2} &= \Omega\sin\alpha\int_0^t dt\\
\tanh^{-1}\frac{\omega_2\left(t\right)}{\Omega\cos\alpha} &= \Omega t\sin\alpha\\
\omega_2\left(t\right) &= \Omega\cos\alpha\tanh\left(\Omega t\sin\alpha\right)
\end{aligned}$$

and

$$\begin{aligned}
\omega_1\left(t\right) = \Omega\cos\alpha\sqrt{1-\tanh^2\left(\Omega t\sin\alpha\right)} &= \frac{\Omega\cos\alpha}{\cosh\left(\Omega t\sin\alpha\right)}\\
\omega_3(t) = \Omega\sin\alpha\sqrt{1-\tanh^2\left(\Omega t\sin\alpha\right)} &= \frac{\Omega\sin\alpha}{\cosh\left(\Omega t\sin\alpha\right)}
\end{aligned}$$