Physics 106b/196b – Problem Set 10 – Due Jan 26, 2007
Solutions

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Problem 1

PROBLEM 1. Ellipsoid of constant Density.

Definition of ellipsoid:

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1;$$

We need to calculate inertia tensor:

$$I_{ij} = \int d^3r\, \rho(\vec{s})\left(s^2\delta_{ij} - s_i s_j\right)$$ which

is equivalent to

(2) $$I_{xx} = \int dx\,dy\,dz\,\rho(\vec{r})(y^2+z^2)$$
$$I_{yy} = \int dx\,dy\,dz\,\rho(\vec{r})(x^2+z^2)$$
$$I_{zz} = \int dx\,dy\,dz\,\rho(\vec{r})(x^2+y^2)$$

and for nondiagonal elements $I_{xy} = -\int dx\,dy\,dz\,\rho(\vec{r})\,xy$ and so on.

→ First method to solve this problem

$$I_{xx} = \rho \cdot \int_{-a}^{a} dx \underbrace{\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}}_{\text{integral over "y"}} \underbrace{\int_{-c\sqrt{1-\frac{y^2}{b^2}-\frac{x^2}{a^2}}}^{c\sqrt{1-\frac{y^2}{b^2}-\frac{x^2}{a^2}}}}_{\text{integral over z}} dy\,dz \cdot (y^2+z^2)$$

This integral can be calculated using computer programs (Mathematica)

Let's do the hardest one of these integrals more explicitly to be clear about it:

$$\begin{aligned}
&\rho \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ z^2 \\
&\quad = \rho \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\ \left[\frac{z^3}{3}\right]\Bigg|_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} \\
&\quad = \frac{2}{3}\, c^3 \rho \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\ \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{3/2}
\end{aligned}$$

You can do this integral by a trigonometric substitution (which is essentially the same as switching to ellipsoidal coordinates as is done in the alternate solution given a couple pages below), look up this integral (*e.g.*, http://www.sosmath.com/tables/integral/integ13/integ13.html), or have Mathematica do it. Looking it up gives

$$\begin{aligned}
&\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\ \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{3/2} \\
&\quad = b \left[\frac{1}{4}\,\frac{y}{b}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2} + \frac{3}{8}\,\frac{y}{b}\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}\right]\Bigg|_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \\
&\qquad + b\left[\frac{3}{8}\left(1 - \frac{x^2}{a^2}\right)^2 \sin^{-1}\left(\frac{y}{b}\frac{1}{\left(1-\frac{x^2}{a^2}\right)^{1/2}}\right)\right]\Bigg|_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \\
&\quad = \frac{3}{4}\, b \left(1 - \frac{x^2}{a^2}\right)^2 \sin^{-1}(1) = \frac{3\,\pi\, b}{8}\left(1 - \frac{x^2}{a^2}\right)^2
\end{aligned}$$

So we have

$$\int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ z^2 = \frac{\pi\, b\, c^3}{4}\int_{-a}^{a} dx\ \left(1 - \frac{x^2}{a^2}\right)^2$$

which can be done using the trigonometric substitution $x = a\sin\theta$, giving

$$\begin{aligned}
\int_{-a}^{a} dx\ \left(1 - \frac{x^2}{a^2}\right)^2 &= a\int_{-\pi/2}^{\pi/2} d\theta\ \cos\theta\left(1 - \sin^2\theta\right)^2 = a\int_{-\pi/2}^{\pi/2} d\theta\ \cos\theta\left(1 - 2\ \sin^2\theta + \sin^4\theta\right) \\
&= a\left[\sin\theta - \frac{2}{3}\ \sin^3\theta + \frac{1}{5}\sin^5\theta\right]\Bigg|_{-\pi/2}^{\pi/2} = 2\,a\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\,a}{15}
\end{aligned}$$

So we have in the end

$$\rho\int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\ z^2 = \left(\frac{\pi\, b\, c^3 \rho}{4}\right)\left(\frac{16\,a}{15}\right) = \frac{4}{15}\,\pi\, a\, b\, c^3 \rho$$

The answer is:

$$I_{xx} = \rho \cdot \frac{4}{15}\pi\left(ab^3c + abc^3\right) = \frac{4}{15}\pi\rho\, abc\left(b^2+c^2\right);$$

Taking into account that volume of ellipsoid is $V = \frac{4}{3}\pi abc$, we have:

$$\boxed{I_{xx} = \frac{1}{5}M\left(b^2+c^2\right)}$$

3) Analogously for I_{yy} and I_{zz} we have:

$$\boxed{I_{yy} = \frac{1}{5}M\left(a^2+c^2\right)}; \quad \boxed{I_{zz} = \frac{1}{5}M\left(a^2+b^2\right)}$$

All "nondiagonal" moments vanish because we have integrals of the type $I_{xy} \sim \int\limits_{-a}^{a} xy\, dx\,dy\,dz$

odd function in symmetric limits.

(4) The answer to the problem is:

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} \frac{M}{5}(b^2+c^2) & 0 & 0 \\ 0 & \frac{M}{5}(a^2+c^2) & 0 \\ 0 & 0 & \frac{M}{5}(a^2+b^2) \end{pmatrix}$$

→ Alternative method

We can avoid calculating those difficult integrals if we define integration variables $\{\delta, \theta, \varphi\}$ as:

$$\begin{cases} x = a\delta\sin\theta\cos\varphi \\ y = b\delta\sin\theta\sin\varphi \\ z = c\delta\cdot\cos\theta \end{cases}$$

In these coordinates equation for the ellipsoid (1) becomes $\boxed{\delta^2 \le 1}$

Limits of integration:

$$0 \le \delta \le 1; \quad 0 \le \theta \le \pi; \quad 0 \le \varphi \le 2\pi;$$

$$d^3V = dx\cdot dy\cdot dz = abc\,\delta^2\cdot\sin\theta\cdot d\delta\cdot d\theta\cdot d\varphi$$

As an example of calculations in these coordinates we can verify formula for volume:

$$M = \rho\cdot\int d^3V = \rho\, abc \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\pi} \sin\theta\, d\theta}_{\int_{-1}^{1} d\mu = 2} \underbrace{\int_0^1 \delta^2 d\delta}_{\frac{1}{3}} = \underbrace{\frac{4\pi abc}{3}}_{\text{Volume}}\rho$$

In order to calculate the moment of inertia elements, we need integrals.

$\iiint x^2 dx\,dy\,dz$, $\iiint y^2 dx\,dy\,dz$, $\iiint z^2 dx\,dy\,dz$

Consider one of them:

$$\iiint z^2 dx\,dy\,dz = abc^3 \int_0^{2\pi} d\varphi \underbrace{\int_0^{\pi} \cos^2\vartheta \cdot \sin\vartheta \cdot d\vartheta}_{= \int_{-1}^{1} \mu^2 d\mu = \frac{2}{3}} \cdot \underbrace{\int_0^1 \rho^4 d\rho}_{= \frac{1}{5}} \Rightarrow = \frac{4\pi abc}{15} c^2 = \frac{M}{5} c^2$$

(5) Analogously,

$$\boxed{\iiint x^2 dx\,dy\,dz = \frac{M}{5} a^2}, \quad \boxed{\iiint y^2 dx\,dy\,dz = \frac{M}{5} b^2}$$

Off-diagonal components vanish because

$$0 = \iiint xy\,dx\,dy\,dz = \iiint yz\,dx\,dy\,dz = \iiint xz\,d^3V$$

(Region of integration is symmetric under inversion $x \to -x$ {or $y \to -y$, $z \to -z$}

Integral of a function odd in x or y turns 0.

Combining results (5), we arrive at the same result

Problem 2

We assume $\widehat{z}$ along the bar in the body frame F and $\widehat{z}'$ along $\vec{\omega}$ in the body frame F'. At $t=0$, $\widehat{x}$ is along $\widehat{x}'$ and the bar stays in the $y'z'$ plane. The angular velocity is

$$\vec{\omega} = \widehat{z}' \frac{v}{\frac{l}{2}\sin\theta}$$

(Note: no underline or prime symbol is needed on $\vec{\omega}$ in the above because it is being written in terms of another vector: the above equation holds in any frame. The components of $\widehat{z}'$ of course depend on the frame: in F', the coordinate representation of $\widehat{z}'$ is $(0,0,1)$; in F, as we indicate below, $\widehat{z}'$ has coordinate representation $(0, \sin\theta, \cos\theta)$. Similary, if we had written $\vec{\omega}$ out in component form, as $\left(0, 0, \frac{v}{\frac{l}{2}\sin\theta}\right)$, then we would have had to specify a representation using the underline and prime. These are subtle distinctions but ones that are importnat to understand.) The decomposition for $\widehat{z}'$ is

$$\widehat{z}' = \widehat{y}\sin\theta + \widehat{z}\cos\theta$$

since $\vec{\omega}$ always stays in the yz plane. So we have

$$\vec{\omega} = \frac{2v}{l\sin\theta}\left(\widehat{y}\sin\theta + \widehat{z}\cos\theta\right)$$

(Refer above to why no underline is needed.) The representation of the torque in F is given by the Euler's equation by

$$\underline{\overrightarrow{\tau}} = \frac{d}{dt}\underline{\overrightarrow{L}} + \underline{\vec{\omega}} \times \underline{\overrightarrow{L}}$$

where

$$\underline{\overrightarrow{L}} = \underline{\mathcal{I}}\,\underline{\vec{\omega}} = \begin{pmatrix} \frac{ml^2}{2} & 0 & 0 \\ 0 & \frac{ml^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2v}{l} \\ \frac{2v\cos\theta}{l\sin\theta} \end{pmatrix} = \begin{pmatrix} 0 \\ mlv \\ 0 \end{pmatrix}$$

(The underlines were needed in the torque equation because the underlines imply whether the time derivative is with respect to the space or body frame. Underlines were needed in the equation for $\underline{L}$ because now one is writing the components.) The components of $\underline{\vec{L}}$ are constant, so the time derivative vanishes. In vector form, it holds that

$$\vec{L} = m\,l\,v\,\hat{y}$$

Understand, again, that now we need not use underlines because $\vec{L}$ is being written in terms of another vector, not in component form $(0, m\,l\,v, 0)$. We are left with

$$\overrightarrow{\tau} = \vec{\omega} \times \overrightarrow{L} = -\widehat{x}\frac{2v\cos\theta}{l\sin\theta}mlv = -\widehat{x}\frac{2mv^2\cos\theta}{\sin\theta}$$

In the above, we no longer need underlines because there is no longer a frame-dependent time derivative. For $\widehat{x}$, we get

$$\begin{aligned} \widehat{x} &= \widehat{x}'\cos\omega t + \widehat{y}'\sin\omega t \\ &= \widehat{x}'\cos\left(\frac{2vt}{l\sin\theta}\right) + \widehat{y}'\sin\left(\frac{2vt}{l\sin\theta}\right) \end{aligned}$$

and therefore

$$\overrightarrow{\tau} = -\frac{2mv^2\cos\theta}{\sin\theta}\left(\widehat{x}'\cos\left(\frac{2vt}{l\sin\theta}\right) + \widehat{y}'\sin\left(\frac{2vt}{l\sin\theta}\right)\right)$$

Again, because $\vec{\tau}$ is being written in terms of other vectors, no underlines or primes are needed. Of course, the coefficients of $\widehat{x}'$ and $\widehat{y}'$ are the components of the F' representation, so it's easy to see what you would write down for $\underline{\vec{\tau}}'$:

$$\underline{\vec{\tau}}' = -\frac{2mv^2\cos\theta}{\sin\theta}\begin{pmatrix}\cos\left(\frac{2vt}{l\sin\theta}\right)\\ \sin\left(\frac{2vt}{l\sin\theta}\right)\\ 0\end{pmatrix}$$

We can also to calculate $\underline{\overrightarrow{\tau}}'$ from first principles. First we need to obtain $\underline{\mathcal{I}}'$ first. The coordinates for two mass points in F' are

$$\underline{\vec{r}}'_1 = \frac{l}{2}\left(-\sin\theta\sin\omega t, \sin\theta\cos\omega t, \cos\theta\right)$$
$$\underline{\vec{r}}'_2 = \frac{l}{2}\left(\sin\theta\sin\omega t, -\sin\theta\cos\omega t, -\cos\theta\right)$$

(Note the underlines are needed because we are writing out components, not expanding in terms of unit vectors.) So

$$\underline{\mathcal{I}}' = \frac{ml^2}{2}\begin{pmatrix}1-\sin^2\theta\sin^2\omega t & -\sin^2\theta\sin\omega t\cos\omega t & -\sin\theta\cos\theta\sin\omega t\\ -\sin^2\theta\sin\omega t\cos\omega t & 1-\sin^2\theta\cos^2\omega t & \sin\theta\cos\theta\cos\omega t\\ -\sin\theta\cos\theta\sin\omega t & \sin\theta\cos\theta\cos\omega t & \sin^2\theta\end{pmatrix}$$

and

$$\begin{aligned}\underline{\overrightarrow{L}}' &= \underline{\mathcal{I}}'\underline{\vec{\omega}}'\\ &= \frac{ml^2}{2}\begin{pmatrix}1-\sin^2\theta\sin^2\omega t & -\sin^2\theta\sin\omega t\cos\omega t & -\sin\theta\cos\theta\sin\omega t\\ -\sin^2\theta\sin\omega t\cos\omega t & 1-\sin^2\theta\cos^2\omega t & \sin\theta\cos\theta\cos\omega t\\ -\sin\theta\cos\theta\sin\omega t & \sin\theta\cos\theta\cos\omega t & \sin^2\theta\end{pmatrix}\begin{pmatrix}0\\0\\ \omega\end{pmatrix}\\ &= \frac{ml^2\omega}{2}\begin{pmatrix}-\sin\theta\cos\theta\sin\omega t\\ \sin\theta\cos\theta\cos\omega t\\ \sin^2\theta\end{pmatrix}\end{aligned}$$

Finally we have

$$\begin{aligned}\underline{\overrightarrow{\tau}}' = \frac{d}{dt}\underline{\overrightarrow{L}}' &= -\frac{ml^2\omega^2}{2}\begin{pmatrix}\sin\theta\cos\theta\cos\omega t\\ \sin\theta\cos\theta\sin\omega t\\ 0\end{pmatrix}\\ &= -\frac{ml^2\omega^2}{2}\begin{pmatrix}\sin\theta\cos\theta\cos\left(\frac{2vt}{l\sin\theta}\right)\\ \sin\theta\cos\theta\sin\left(\frac{2vt}{l\sin\theta}\right)\\ 0\end{pmatrix}\\ &= -\frac{2mv^2\cos\theta}{\sin\theta}\begin{pmatrix}\cos\left(\frac{2vt}{l\sin\theta}\right)\\ \sin\left(\frac{2vt}{l\sin\theta}\right)\\ 0\end{pmatrix}\end{aligned}$$

where we had to use underlines and primes in the first line above because of the time derivative.

Problem 3

Solution for 8.25

We start from formula (8.124) on page 322.

$$E' = \frac{I_1}{2}\left(\dot{\theta}^2 + \frac{(L_z - L_3\cdot\cos\theta)^2}{I_1^2\cdot\sin^2\theta}\right) + \frac{\Omega^2}{2}\epsilon I_1\cdot\left(\frac{1-3\cos^2\theta}{2}\right)$$

We introduce $\tilde{L}_z = \frac{L_z}{I_1\Omega}$; $\tilde{L}_3 = \frac{L_3}{I_1\Omega}$;

$\tilde{E}' = \frac{E'}{I_1\Omega^2}$ and we substitute all these new variables into (8.124).

$$\frac{E'}{I_1\Omega^2} = \frac{1}{2}\cdot\frac{1}{\Omega^2}\cdot\left(\frac{I_1^2\dot{\theta}^2\cdot\sin^2\theta + (\tilde{L}_z - \tilde{L}_3\cos\theta)^2 I_1^2\Omega^2}{I_1^2\cdot\sin^2\theta}\right)$$

$$+\frac{\epsilon}{2}\cdot\frac{I_1\Omega^2}{I_1\Omega^2}\cdot\left(\frac{1-3\cdot\cos^2\theta}{2}\right);$$

Note that $\cos\theta \equiv u$; then $\dot{u} = -\dot{\theta}\cdot\sin\theta$.

Note also that we need to express time in units of $\frac{1}{\Omega}$ $\Rightarrow$

$\Rightarrow$ we have instead of $I_1^2\dot{\theta}^2\sin\theta$ modified expression $I_1^2\Omega^2\dot{\theta}^2\sin\theta$.

$\left\{\text{because } \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{d\left(\frac{t}{\Omega}\right)} = \Omega\cdot\frac{d\theta}{dt} = \Omega\cdot\dot{\theta}\right\}$

So, we arrive at

$$\tilde{E}' = \frac{1}{2}\left(\frac{\dot{u}^2 + (\tilde{L}_z - \tilde{L}_3\cdot u)^2}{1-u^2}\right) + \frac{\epsilon}{2}\,\frac{1-3u^2}{2}$$

Remove tildas $\Rightarrow$ instead of formula (8.137) in the book we arrive at

$$\boxed{E' = \frac{1}{2}\left(\frac{\dot{u}^2 + (L_z - L_3 u)^2}{1-u^2}\right) + \frac{\epsilon}{4}(1-3u^2)}$$

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There is a mistake in Hand & Finch

Part b. Estimation for L_z, L_3, E'_{min}.

We start from the formula (8.122) and (8.123):

$$\begin{cases} P_\psi \equiv L_3 = I_3 \cdot (\dot{\psi} + \dot{\phi} \cdot \cos\theta) \\ P_\phi \equiv L_z = I_1 \cdot \dot{\phi} \cdot \sin^2\theta + I_3 \cdot (\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \end{cases}$$

$\cos\theta \equiv u$ and $\dot{\phi} \sim 7 \cdot 10^{-12} \frac{\text{Rad}}{\text{sec}}$ according to eq. (8.121)

$\dot{\psi}$ is just the rotation speed of the earth. But, be careful, it does not correspond simply to the length of the solar day; recall that the sidereal day, the length of time needed for the earth to complete one rotation relative to the fixed stars, is 23.9344696 hours or 86164.1 seconds, so

$$\dot{\psi} = \frac{2\,\pi}{86164.1 sec} = 7.29212 \times 10^{-5} \text{ rad/sec}$$

We want to express time in units of $1/\Omega$, so we note that, from Equation (8.116),

$$\Omega = \sqrt{12.446 \times 10^{-14} \text{ rad/sec}^2} = 3.52789 \times 10^{-7} \text{ rad/sec}$$

Next, we want to calculate angular momenta, so we will need the ratio I_3/I_1. The problem gives $\epsilon = \frac{1}{305.8}$ where $\epsilon \equiv \frac{I_3 - I_1}{I_1}$ as defined on p. 321 just above Equation (8.118). This value of ϵ is, for unknown reasons, different than the value given in the same place as ϵ is defined on p. 321, which is $\epsilon = 0.00335281 = 298.3$. We blindly use the value given in the problem. This implies

$$\frac{I_3}{I_1} = 1 + \epsilon = 1.00327$$

Next, we want L_z. Formally, $L_z = I_3 \left(\dot{\psi} + \dot{\phi}\, \cos\theta\right)$. Since $\dot{\phi}/\dot{\psi} = \mathcal{O}(10^7)$, we can ignore $\dot{\phi}$ given the precision we are aiming for in the result (6 significant digits, or 1 part in 10^6). So we have simply

$$L_3 \approx \frac{I_3 \dot{\psi}}{I_1 \, \Omega} = 1.00327 \cdot \frac{7.29212 \, \times \, 10^{-5} \text{ rad/sec}}{3.52789 \, \times \, 10^{-7} \text{ rad/sec}} = 207.38$$

This differs from Hand and Finch's result, we believe their result is incorrect. Next, we want L_z, which is formally $L_z = I_1 \, \dot{\phi}^2 \, \sin^2 \theta + L_3 \, \cos \theta$. Again, though, $\dot{\phi}$ is small compared $\dot{\psi}$, and $I_1 \approx I_3$, so we can drop the first term, leaving

$$L_z = L_3 \, \cos \theta = 207.38 \cdot \cos 23.45^\circ = 190.247$$

which, again is different from the result given in Hand and Finch. They do seem to be using the same θ, though, because the ratio $L_z / L_3 = \cos \theta = 0.9174$ for both our result and theirs. Finally, we want to calculate E'_{min}. Since $L_z = L_3 \, \cos \theta = L_3 \, u$ to very good approximate, and $\dot{u} = 0$ by definition of what is mean by E'_{min}, it holds that the first term in E' vanishes for $E' = E'_{min}$. So we have

$$E'_{min} = \frac{\epsilon}{4} \left(1 - 3 \, u^2\right) = -0.00124666$$

which is exactly a factor of 2 smaller than the result given in Hand and Finch because of their factor of 2 error in the expression for the second term.

Part (c). Using results for L_3 and L_z, and using the fact that $\boxed{\dot{u} = 0}$ at the turning point (see page 315 in Hand-Finch):

$$E'(\theta = 22.24^\circ) = \frac{1}{2} \cdot \left(\frac{L_z - L_3 \cdot \cos(22.24^\circ)}{\sin^2(22.24^\circ)}\right)^2 + \frac{\epsilon}{4} \cdot \left(1 - 3\cos^2(22.24^\circ)\right) \approx 10$$

$E'(\theta = 24.73^\circ) \cong 10$ also

Part (d) We need to solve for $\dot{u}$:

$$\dot{u}^2 + (L_z - L_3 u)^2 = 2(1-u^2)\left(E' - \frac{\epsilon}{4}(1-3u^2)\right)$$

$$\dot{u}^2 = 2(1-u^2)\left(E' - \frac{\epsilon}{4}(1-3u^2)\right) - (L_z - L_3 u)^2;$$

$$\dot{u}(t) = \sqrt{2(1-u^2) \cdot \left[E' - \frac{\epsilon}{4}(1-3u^2)\right] - (L_z - L_3 u)^2};$$

Period is

$$\boxed{\tau = 2 \cdot \int_{\cos(24.73^\circ)}^{\cos(22.24^\circ)} du \cdot \frac{1}{\sqrt{2(1-u^2) \cdot \left[E' - \frac{\epsilon}{4}(1-3u^2)\right] - (L_z - L_3 u)^2}}}$$

where one would insert the numerical values $E' = 10$, $L_3 = 207.38$, and $L_z = 190.247$ and integrate numerically over u to obtain τ.

Problem 4

One can do this problem in two ways. The first uses the instantaneous point of contact of the coin with the floor as the origin of the rotating coordinate system. The second uses the CM. We explicitly write out the first version, then describe why the second version yields the same result.

(a) The angular velocity of the coin is

$$\vec{\omega} = \omega_p\, \hat{z}' + \Omega\, \hat{z}$$

There are two ways to see what the condition of rolling without slipping implies. The first is by analogy to Problem 4 from Problem Set 9, the cone rolling on the plane. There, we related the path length that the edge of the cone must travel for a single period τ of the precession motion. Applied here, let τ again be the precession period, $\tau = \frac{2\,\pi}{\omega_p}$. The circumference of the circle that the edge of the coin travels on is $2\,\pi\,R\,\cos\theta$. Rolling without slipping implies that ψ angle of the coin must rotate through an angle $2\,\pi\,\frac{2\,\pi\,R\,\cos\theta}{2\,\pi\,R}$ in this time τ. So Ω, which is given by $\dot{\psi}$, is just $\frac{2\,\pi}{\tau}\,\frac{2\,\pi\,R\,\cos\theta}{2\,\pi\,R} = \omega_p\,\cos\theta$. One can see empirically that a negative sign is needed because ψ rotates backward if the precession is forward in angle, so really

$$\Omega = -\omega_p\,\cos\theta$$

(b) The F_P systems is defined to precess with the coin's 3-axis but not spin with it. Therefore, $\widehat{z}'$ always stays in the $y_P z_P$ plane and can be decomposed into

$$\widehat{z}' = \widehat{z}_P\cos\theta - \widehat{y}_P\sin\theta$$

The angular velocity of the coin is

$$\begin{aligned}\vec{\omega} &= \omega_p\widehat{z}' + \Omega\widehat{z}_P \\ &= \omega_p\left(\widehat{z}_P\cos\theta - \widehat{y}_P\sin\theta\right) + \Omega\widehat{z}_P \\ &= -\widehat{y}_P\omega_p\sin\theta\end{aligned}$$

(c) In F_P, the new moment of inertia tensor relative to the point on the edge of the coin contact with the surface is

$$\begin{aligned}\underline{\mathcal{I}}_d &= \underline{\mathcal{I}} + M\left(l^2\mathbf{1} - \vec{l}\,\vec{l}^{\,T}\right) \\ &= \begin{pmatrix}\frac{1}{4}MR^2 & 0 & 0 \\ 0 & \frac{1}{4}MR^2 & 0 \\ 0 & 0 & \frac{1}{2}MR^2\end{pmatrix} + M\begin{pmatrix}R^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R^2\end{pmatrix} \\ &= \begin{pmatrix}\frac{5}{4}MR^2 & 0 & 0 \\ 0 & \frac{1}{4}MR^2 & 0 \\ 0 & 0 & \frac{3}{2}MR^2\end{pmatrix}\end{aligned}$$

where $\vec{l} = R\widehat{y}_P$. So the angular the momentum is

$$\underline{\vec{L}}_P = \underline{\mathcal{I}}_d\,\underline{\vec{\omega}} = -\frac{1}{4}MR^2\omega_p\sin\theta\begin{pmatrix}0\\1\\0\end{pmatrix} = -\frac{1}{4}MR^2\omega_p\sin\theta\widehat{y}_P$$

For the torque, we have three forces acting: the force of gravity acting at the CM of the coin, the normal force acting at the edge of the coin, and the frictional force that enforces rolling without slipping, also at the edge of the coin. The latter two act at the origin of our coordinate system, the point of contact with the floor, so they yield no torque. So the torque is simply

$$\begin{aligned} \underline{\vec{\tau}}_P &= \underline{\vec{r}}_P \times \underline{\vec{F}}_P \\ &= MgR\cos\theta\,\widehat{x}_P \end{aligned}$$

where $\underline{\vec{r}}_P = -R\widehat{y}_P$ and $\underline{\vec{F}}_P = -Mg\left(\widehat{z}_P\cos\theta - \widehat{y}_P\sin\theta\right)$. In F_P, Euler's equations are

$$\frac{d}{dt}\underline{\vec{L}}_P = -\underline{\vec{\omega}}_P \times \underline{\vec{L}}_P + \underline{\vec{\tau}}_P$$

From our expression for $\underline{\vec{L}}_P$ above, we see that it is constant. So Euler's equations reduce to

$$\begin{aligned} \underline{\vec{\tau}}_P &= \underline{\vec{\omega}}_P \times \underline{\vec{L}}_P \\ MgR\cos\theta\,\widehat{x}_P &= \omega_p\widehat{z}' \times \left(-\frac{1}{4}MR^2\omega_p\sin\theta\widehat{y}_P\right) \\ MgR\cos\theta\,\widehat{x}_P &= \omega_p\left(\widehat{z}_P\cos\theta - \widehat{y}_P\sin\theta\right) \times \left(-\frac{1}{4}MR^2\omega_p\sin\theta\,\widehat{y}_P\right) \\ MgR\,\widehat{x}_P &= \frac{1}{4}MR^2\omega_p^2\sin\theta\,\widehat{x}_P \\ \omega_p &= 2\sqrt{\frac{g}{R\sin\theta}} \end{aligned}$$

We note that one can solve the problem in an alternate fashion. We chose the point of contact between the coin and the floor as the origin of our coordinate system, so the only torque was due to gravity. Alternatively, we can place the origin of the coordinate system at the center of mass. These coordinate systems are the same except for a displacement $\vec{l} = R\,\hat{y}_P$. The angular velocity is unaffected since the new F_P coordinate system orientation is the same as the one used above. The angular momentum is also unaffected because it depends only on the y moment of inertia, which we saw above is the same for the CM and the displaced versions of the inertia tensor. The only other change we need to worry about is in torque. In the case with the origin at the CM, the gravitational torque is zero, but we will instead have a normal force torque and a frictional torque. We may neglect the frictional torque because it acts along the $\hat{z}$ axis and is simply the force that causes rolling without slipping. The normal force torque is identical to the gravity torque in the above because both the radius vector and the force have sign flips. So the angular velocity, angular momentum, and torque is the unchanged in spite of the change of origin, so the solution for ω_p will be unchanged also.

Problem 5

Problem 5 Routhian

Coordinate q_N is ignorable in initial Lagrangian $L(q_1, q_2, q_3, \dots, q_{N-1}, \dot q_1, \dots, \dot q_N, t)$

Using a Legendre transformation, we construct the Routhian $R = L - p_N \dot q_N$:

$$R(q_1, \dots, q_{N-1}, \dot q_1, \dot q_2, \dots, \dot q_{N-1}) \equiv L - p_N \dot q_N$$

A) Proof that R does not depend on $\dot q_N$:

$$\frac{\partial}{\partial \dot q_N} R = \frac{\partial}{\partial \dot q_N}\left(L - p_N \dot q_N\right) = \underbrace{\frac{\partial L}{\partial \dot q_N}}_{p_N \equiv \frac{\partial L}{\partial \dot q_N}} - p_N = 0$$

For $i < N$, treating R as Lagrangian,

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_i}\right) - \frac{\partial R}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

because $p_n \dot q_n$ does not depend on q_i or $\dot q_i$.

Part B

The Lagrangian for the top is given in eq. (8.93) of Hand and Finch.

$$L = \frac{I_1}{2}\cdot\left(\dot\theta^2 + \dot\phi^2 \cdot \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\psi + \dot\phi\cos\theta\right)^2 - Mgl\cos\theta;$$

The conserved momenta are

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta)$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \dot\phi \sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta = I_1 \dot\phi \cdot \sin^2\theta + p_\psi \cos\theta$$

Routhian is then

$$R(\theta, \dot\theta) = L - p_\psi \dot\psi - p_\phi \dot\phi = \frac{1}{2} I_1 (\dot\theta^2 - \dot\phi^2 \sin^2\theta) - \frac{1}{2} I_3 \cdot (\dot\psi + \dot\phi\cos\theta)^2 - Mgl\cos\theta;$$

it follows from p_ψ and p_ϕ that $\dot\phi^2 = \frac{1}{(I_1 \sin^2\theta)^2}\cdot\left(p_\phi - p_\psi \cos\theta\right)^2$

and $\left(\dot\psi + \dot\varphi\cos\theta\right)^2 = \left(\frac{P_\psi}{I_3}\right)^2$;

So,

$$R(\theta,\dot\theta) = \frac{1}{2} I_1 \dot\theta^2 - \frac{1}{2I_1\sin^2\theta}\cdot\left(P_\varphi - P_\psi\cos\theta\right)^2 - \frac{1}{2I_3}\cdot P_\psi^2 - Mgl\cos\theta;$$

We can introduce (as in the lecture notes) $V_{eff} \equiv \frac{1}{2I_1\sin^2\theta}\cdot\left(P_\varphi - P_\psi\cos\theta\right)^2 + Mgl\cos\theta$,

so $R(\theta,\dot\theta) = \frac{1}{2} I_1 \dot\theta^2 - \frac{1}{2I_3} P_\psi^2 - V_{eff}(\theta)$

Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial R}{\partial\dot\theta}\right) - \frac{\partial R}{\partial\theta} = 0; \quad I_1\cdot\ddot\theta + \frac{\partial V_{eff}(\theta)}{\partial\theta} = 0$$

$\Rightarrow$ We arrive at

$$\boxed{I_1\ddot\theta = -\frac{\partial}{\partial\theta}\left(\frac{1}{2I_1}\cdot\frac{\left(P_\varphi - P_\psi\cos\theta\right)^2}{\sin^2\theta}\right) + Mgl\cdot\sin\theta}$$

Problem 6

A solution for this will be written shortly.