Physics 106b/196b – Problem Set 11 – Due Feb 2, 2007
Solutions

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Problem 1

(a) The ball moves in the primed frame with velocity u' where

$$u' = \frac{\frac{c}{2} + \frac{c}{2}}{1 + \frac{\left(\frac{c}{2}\right)^2}{c^2}} = \frac{4}{5}c$$

If, in the primed frame, we assume the ball passes one end of the stick at $t' = 0$, the ball passes the other end at $t' = \frac{L'}{u'} = \frac{5L'}{4c}$. The separation of the two events is

$$(\Delta x', \Delta t') = \left(L', \frac{5L'}{4c}\right)$$

An observer in the unprimed frame will find the ball passes the stick during

$$\begin{aligned}
\Delta t &= -\gamma\left(\beta \Delta x' - \Delta t'\right) \\
&= -\frac{1}{\sqrt{1 - \left(\frac{c}{2c}\right)^2}}\left(\frac{c}{2c}\Delta x' - \Delta t'\right) \\
&= -\frac{2}{\sqrt{3}}\left(\frac{1}{2}L' - \frac{5L'}{4c}\right) \\
&= \frac{\sqrt{3}}{2}\frac{L'}{c}
\end{aligned}$$

given by the Lorentz Transformation.

(b) No. Because c is the relative speed of two objects and not the speed of an entity. In the unprimed frame, we find the length of stick is

$$L = L'\sqrt{1 - \left(\frac{c}{2c}\right)^2} = \frac{\sqrt{3}}{2}L'$$

So the ball passes the stick during

$$\Delta t = \frac{L}{c} = \frac{\sqrt{3}}{2}\frac{L'}{c}$$

Problem 2

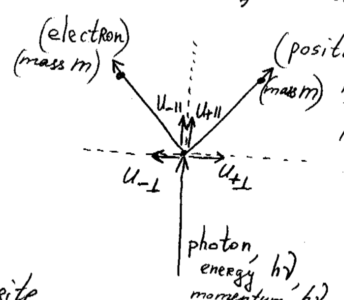
1

Hand & Finch, Problem 12.11 (Photon désintegration)

We will prove, based on the laws of conservation of momentum and energy that a photon cannot disintegrate into an electon (e^-) and a positron (e^+).

In our solution $c=1$.

We introduce components of electron's and positron's velocities



$u_{+\parallel}, u_{-\parallel}$ along the direction of motion of PHOTON

"+" means positron; "−" means electron

$u_{+\perp}, u_{-\perp}$ along the direction orthogonal to the direction of motion of photon.

Let us write the law of conservation of energy (A); law of conservation of momentum in direction of motion of photon (B); and law of conservation of momentum in direction orthogonal to the direction of motion of photon (C)

For this purpose we introduce "total" velocities $\begin{cases} u_+^2 \equiv u_{+\parallel}^2 + u_{+\perp}^2 \\ u_-^2 = u_{-\parallel}^2 + u_{-\perp}^2 \end{cases}$

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$$\begin{cases} (A)\quad h\nu = \gamma_+ m + \gamma_- m \,; \\ (B)\quad h\nu = \gamma_+ m u_{+\parallel} + \gamma_- m u_{-\parallel} \,; \\ (C)\quad 0 = \gamma_+ m u_{+\perp} + \gamma_- m u_{-\perp} \,; \end{cases} \quad \left(\text{where } \gamma_+ \equiv \frac{1}{\sqrt{1-\frac{u_+^2}{c^2}}} \quad \gamma_- \equiv \frac{1}{\sqrt{1-\frac{u_-^2}{c^2}}} \right)$$

1) It follows from (C) that $\gamma_+ m u_{+\perp} = -\gamma_- m u_{-\perp}$

or $\gamma_+^2 u_{+\perp}^2 = \gamma_-^2 u_{-\perp}^2$.

We use the hint that was given in the problem: $\boxed{u^2 = 1 - \frac{1}{\gamma^2}}$

"Total" velocity $u_+^2 = u_{+\parallel}^2 + u_{+\perp}^2$; $u_-^2 = u_{-\parallel}^2 + u_{-\perp}^2$;

(It can be easily proven: $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \rightarrow \gamma^2 \cdot \left(1 - \frac{u^2}{c^2}\right) = 1$,

$\gamma^2 \frac{u^2}{c^2} = \gamma^2 - 1$;

$u^2 = c^2\left(1 - \frac{1}{\gamma^2}\right)$,

or putting $c \equiv 1 \Rightarrow u^2 = 1 - \frac{1}{\gamma^2}$)

Therefore, we have

$$\gamma_+^2 \cdot \left(1 - \frac{1}{\gamma_+^2} - u_{+\parallel}^2\right) = \gamma_-^2 \left(1 - \frac{1}{\gamma_-^2} - u_{-\parallel}^2\right);$$

$$\gamma_+^2 - 1 - \gamma_+^2 u_{+\parallel}^2 = \gamma_-^2 - 1 - \gamma_-^2 u_{-\parallel}^2 \,;$$

$$\boxed{\gamma_+^2 \cdot \left(1 - u_{+\parallel}^2\right) = \gamma_-^2 \left(1 - u_{-\parallel}^2\right)} \;;$$

③

2) We see that Equations ⓐ and ⓑ both contain $h\nu$ in their left-hand side so their right-hand sides are equal to each other:

$$\gamma_+ m + \gamma_- m = \gamma_+ m u_{+\parallel} + \gamma_- m u_{-\parallel};$$

$$\gamma_+ + \gamma_- = \gamma_+ u_{+\parallel} + \gamma_- u_{-\parallel};$$

Squaring this equation, we arrive at:

$$\gamma_+^2 + \gamma_-^2 + 2\gamma_+\gamma_- = \gamma_+^2 u_{+\parallel}^2 + \gamma_-^2 u_{-\parallel}^2 + 2\gamma_+\gamma_- u_{+\parallel} u_{-\parallel};$$

which is equivalent to

$$\underbrace{\gamma_+^2(1-u_{+\parallel}^2) + \gamma_-^2(1-u_{-\parallel}^2)}_{\text{positive value} > 0} = 2\gamma_+\gamma_- \underbrace{(u_{+\parallel}u_{-\parallel} - 1)}_{\text{negative value} < 0};$$

We arrive at the contradiction which means that we cannot satisfy simultaneously to the laws of conservation of energy and momentum. Therefore, disintegration of a photon into electron and positron is impossible.

④

→ Alternative (probably easier to interpret) way of solving part (A)

Using the same equations (a) and (b) for the (a) energy conservation and (b) momentum along the direction of photon propagation conservation, we see that

$$(b')\ h\nu = p_{+\parallel} + p_{-\parallel}; \quad \text{where } \begin{cases} p_{+\parallel} = \gamma_+ m u_{+\parallel}, \\ p_{-\parallel} = \gamma_- m u_{-\parallel}. \end{cases}$$

$$(A')\ (h\nu)^2 = E_+^2 + E_-^2 + 2E_+E_-;$$

Introducing momentum $p_{+\perp}$ and $p_{-\perp}$ in orthogonal direction, i.e. $p_{+\perp} = \gamma_+ m u_{+\perp}$ and $p_{-\perp} = \gamma_- m u_{-\perp}$, we can rewrite (b') as $(h\nu)^2 = p_{+\parallel}^2 + p_{-\parallel}^2 + 2p_{+\parallel}p_{-\parallel}$;

It follows from eq. (c) that

$$|p_{+\perp}| = |p_{-\perp}|, \quad \text{and} \begin{cases} p_+^2 = p_{+\parallel}^2 + p_{+\perp}^2; \\ p_-^2 = p_{-\parallel}^2 + p_{-\perp}^2; \end{cases}$$

We also use $E_+ = \sqrt{m^2 + p_+^2}$; $E_- = \sqrt{m^2 + p_-^2}$;

(5)

Substituting into (A'), we have:

$$E_+^2 + E_-^2 + 2E_+E_- =$$

$$= \left(\sqrt{m^2+p_+^2}\right)^2 + \left(\sqrt{m^2+p_-^2}\right)^2 +$$

$$+ 2\sqrt{m^2+p_+^2}\cdot\sqrt{m^2+p_-^2} =$$

$$= \boxed{p_+^2 + p_-^2 + 2m^2 + 2\sqrt{m^2+p_+^2}\sqrt{m^2+p_-^2}} =\!=$$

using the fact that $(a')=(b')^2$

$$= p_{+\parallel}^2 + p_{-\parallel}^2 + 2\cdot p_{+\parallel}\cdot p_{-\parallel} =$$

$$= \left(p_+^2 - p_{+\perp}^2\right) + \left(p_-^2 - p_{-\perp}^2\right) + 2\cdot\sqrt{p_+^2 - p_{+\perp}^2}\sqrt{p_-^2 - p_{-\perp}^2}$$

$$= \boxed{p_+^2 + p_-^2 - 2p_{+\perp}^2 + 2\sqrt{p_+^2 - p_{+\perp}^2}\cdot\sqrt{p_-^2 - p_{-\perp}^2}};$$

Comparing the equations in the wavy frames,

(6)

which <u>must be equal</u> to each other, because they represent the fact that $(E_+ + E_-)^2 = (p_{+\parallel} + p_{-\parallel})^2 = (h\nu)^2$,

we can argue that they <u>cannot be equal</u> to each other!

Indeed, the upper equation has the structure $p_+^2 + p_-^2 + 2\sqrt{p_+^2 + \text{positive Number}}\sqrt{p_-^2 + \text{positive Number}} +$

$+ 2m^2$ Again positive Number

(7)

while the lower expression has the structure

$$p_+^2+p_-^2+2\sqrt{p_+^2-\text{positive Number}}\cdot\sqrt{p_-^2-\text{positive Number}}-\underbrace{2p_{+\perp}^2}_{\text{positive number}}$$

but we subtract it!

So, we have that the upper expression in the frame on the previous page can be represented as $p_+^2+p_-^2+\Delta_1+\Delta_3$ while the lower expression as $p_+^2+p_-^2+\Delta_2-\Delta_4$

Where $\Delta_2<\Delta_1$, and all $\Delta_3, \Delta_4, \Delta_1, \Delta_2$ are positive.

This contradiction shows that a photon cannot decay into a positron and electron.

Part (b)

(8)

We will show in this part of the problem that if a stationary nucleus of mass M is present nearby, then under some condition, disintegration of a photon into electron and positron becomes possible.

Law of conservation of energy now becomes:

$$\underbrace{h\nu+Mc^2}_{\substack{\text{the nucleus was stationary}\\ \text{before the disintegration of photon}\\ \text{Rest energy} = Mc^2\\ \text{Momentum} = 0}}=\underbrace{M\gamma c^2}_{\substack{\text{The nucleus is moving with some finite speed } V\\ \text{after the photon disintegration,}\\ \text{so the energy of nucleus is now } \frac{M}{\sqrt{1-\frac{V^2}{c^2}}}c^2 = M\gamma c^2}}+\gamma_+mc^2+\gamma_-mc^2$$

Again, putting $c=1$, we have:

$$\begin{cases} h\nu+M=M\gamma+\gamma_+m+\gamma_-m\,; \\ h\nu+0=MV_{\parallel}\gamma_{Nucleus}+\gamma_+mu_{+\parallel}+\gamma_-mu_{\parallel}\,; \\ 0=MV_{\perp}\gamma_{Nucleus}+\gamma_+mu_{+\perp}+\gamma_-mu_{-\perp}\,; \end{cases}$$

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Probably, it is possible to carry out transformations analogous to those described on pages 2-3 of solution of part (A), but I will argue that now we do not arrive to a contradiction, using the second (alternative) way of solving problem (A), described on pages (4)-(7).

We saw that if we start from equations

$$\begin{cases} h\nu = P_{+\parallel} + P_{-\parallel} \\ h\nu = E_+ + E_- \end{cases} \Rightarrow \text{we arrive to a contradiction that}$$

$$(E_+ + E_-)^2 > (P_{+\parallel} + P_{-\parallel})^2$$

although it should be $(h\nu)^2 = (E_+ + E_-)^2 = (P_{+\parallel} + P_{-\parallel})^2$;

Now, if there is a nucleus of the rest mass M, we start from the equations

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$$\begin{cases} h\nu + M = M\gamma + E_+ + E_- \,; \\ h\nu = \underbrace{M \cdot V_\parallel \gamma}_{\text{Nucleus}} + \underbrace{P_{\parallel +}}_{\text{positron}} + \underbrace{P_{\parallel -}}_{\text{electron}} \,; \end{cases}$$

Therefore, now we have

$$(h\nu)^2 = \begin{cases} \xrightarrow{\text{From the energy equation}} \left(E_+ + E_- + M(\gamma - 1)\right)^2 \\ \xrightarrow{\text{from the momentum equation}} \left(P_{+\parallel} + P_{-\parallel} + M V_\parallel \gamma\right)^2 \end{cases},$$

and now we increase the second term for some choice of masses M; and therefore we could have equality now

$$\left(E_+ + E_- + M(\gamma - 1)\right)^2 = \left(P_{+\parallel} + P_{-\parallel} + M V_\parallel \gamma\right)^2.$$

Let us find minimum energy $h\nu$ of photon so that the disintegration is possible.

(11)

Minimum energy corresponds to the case that both positron and electron have no kinetic energy.

The equations of energy and momentum conservation become:

$$\begin{cases} h\nu + M = M\gamma_N + 2m_e \; ; & \text{only rest-energy} \\ h\nu + 0 = MV_{\parallel}\gamma_N + 0 + 0 \; ; \\ 0 = MV_{\perp}\gamma_N + 0 + 0 \; ; \Rightarrow V_{\perp} = 0 \end{cases}$$

Consider the second equation here: (velocity for nucleus)

$$V_{\parallel}\cdot\gamma_N = V_{\parallel}\cdot\frac{1}{\sqrt{1-V_{\parallel}^2}} \; ; \quad \gamma_N = \frac{1}{\sqrt{1-V_{\parallel}^2}} \; ; \quad \gamma_N^2 - \gamma_N^2 V_{\parallel}^2 = 1$$

$$V_{\parallel}^2\gamma_N^2 = \gamma_N^2 - 1 \; ;$$

$$\Rightarrow (h\nu)^2 = M^2(\gamma_N^2 - 1) \; ; \Rightarrow \underline{\gamma_N^2 = \frac{(h\nu)^2}{M^2} + 1} \; ;$$

Substitute this γ_N^2 into the first equation:

(12)

$$\left[h\nu + M - 2m_e\right]^2 = M^2\cdot\gamma_N^2 = (h\nu)^2 + 1\cdot M^2 \; ;$$

Denoting $h\nu \equiv E \Rightarrow$

$$\cancel{E^2} + 2E\cdot(M - 2m_e) + (M - 2m_e)^2 = \cancel{E^2} + 1\cdot M^2 \; ;$$

$$2E\cdot(M - 2m_e) + \cancel{M^2} - 4M\cdot m_e + 4m_e^2 = \cancel{M^2} \; ;$$

$$\boxed{E = 2\left(\frac{M - m_e}{M - 2m_e}\right)\cdot m_e \approx (\text{as } M \gg m_e) \approx 2m_e c^2 .}$$

Some additional comments are warranted:

- What's really going on in part (a)? The fundamental problem is that it's just impossible to conserve energy and momentum in going from a single photon to a final state with any non-zero-rest-mass particles. That's all. What we have done is to demonstrate this in different ways by assuming it is possible and then obtaining a contradiction. Another way of seeing this is to look at $E/|\vec{p}|$ in a general way. For photons, this quantity is always 1. For massive particles, this quantity is $1/\beta$, which is always greater than 1. There is thus no way to make up a four-momentum from massive particles that is consistent with a four-momentum made up of massless particles.

- How does this change with the presence of a nucleus? The nucleus has rest energy but zero spatial momentum. This breaks the tight tie between the energy and spatial momentum in the initial state, which is what makes simultaneous conservation of energy and momentum impossible if there are non-zero-rest-mass particles. In the language of the previous bullet point, introducing M in the initial state makes $E/|\vec{p}| > 1$ for the initial state also.

- Why is the final state with the electron and positron created at rest the lowest energy final state? It is a somewhat subtle argument. Clearly, the absolute minimum photon energy consistent with just the final state particle count is a photon with energy $2\,m_e$ so that there is enough energy to create the final state electron and positron. But we can't have a final state in which M and the positron and electron are at rest – such a state has no spatial momentum, while the initial state has spatial momentum. So, how can we get the necessary momentum while minimimally increasing the final state energy? By giving the momentum to the nucleus: because it is so heavy, one doesn't need to give it much speed to get the spatial momentum $\gamma\, m\, \beta$ to be large enough to provide the necessary spatial momentum. This justifies the guess used above. In the end, it is justified by the fact that

$$E = 2\,m_e \left(\frac{M - m_e}{M - 2\,m_e}\right) \approx 2\,m_e \left(1 + \frac{m_e}{M}\right)$$

  That is, E approaches the minimum possible value it can have based on final state particle count as $m_e/M \to 0$; you can't do any better than that.

Problem 3

(a) Since the distant person and the Earth have the same rest frame. This part is easy. In the Earth frame, if we assume tachyons leave the earth at $t' = 0$, then they arrive at the distance person at $t' = \frac{L'}{v_t}$ and return to the earth at $t' = \frac{2L'}{v_t}$. So we have

$$\Delta t' = \frac{2L'}{v_t}$$

(b) Just as part (a), tachyons arrive at the distant person at $t' = \frac{L'}{v_t}$. After the reply is sent, tachyons travel at the velocity of v_t relative to the receiving distant person who moves with a velocity v_0 away from the Earth. Let us assume the distant person is traveling away from the earth with velocity $-v_0$ along the $-\hat{x}$ axis. So the tachyons travel at the velocity of v_t' in the $+\hat{x}$ direction in the Earth frame where

$$v_t' = \frac{v_t - v_0}{1 - \frac{v_t v_0}{c^2}}$$

given by the addition of the velocities. The tachyons reach the Earth at $t' = \frac{L'}{v_t} + \frac{L'}{v_t'}$. So one has

$$\Delta t' = \frac{L'}{v_t} + \left(1 - \frac{v_t v_0}{c^2}\right) \frac{L'}{v_t - v_0}$$

(c) If the Earth-bound observer receive the reply before the original tachyons, we have $\Delta t' < 0$ in part (b). So we have

$$\begin{aligned}
\frac{L'}{v_t} + \left(1 - \frac{v_t v_0}{c^2}\right) \frac{L'}{v_t - v_0} &< 0 \\
\frac{v_t v_0}{c^2} - 1 &> \frac{v_t - v_0}{v_t} \\
v_0 \left(\frac{v_t}{c^2} + \frac{1}{v_t}\right) &> 2 \\
v_0 &> \frac{2v_t}{1 + \frac{v_t^2}{c^2}}
\end{aligned}$$

Problem 4

①

Hand & Finch, Problem 12.12, Part (A)

Decay of B^0-meson $\underbrace{B^0}_{\text{at rest}} \to \frac{J}{\psi} + K^0$

$\vec{p}_B = \vec{p}_J + \vec{p}_K$; $\boxed{c=1}$

$$\begin{cases} m_B = \gamma_J m_J + \gamma_K m_K; & \text{energy conservation law} \\ 0 = \gamma_J m_J \vec{u}_J + \gamma_K m_K \vec{u}_K; & \text{momentum conservation law in the center of mass frame} \end{cases}$$

$\gamma_J m_J \vec{u}_J = -\gamma_K m_K \vec{u}_K$;

Squaring this identity, we get: $\gamma_J^2 m_J^2 u_J^2 = \gamma_K^2 m_K^2 u_K^2$;

We again use the identity that was proven in Problem 12.11 of Hand & Finch: $\gamma^2 = \frac{1}{1-u^2} \Rightarrow \boxed{u^2 = 1 - \frac{1}{\gamma^2}}$

$$\gamma_J^2 m_J^2 \left(1 - \frac{1}{\gamma_J^2}\right) = \gamma_K^2 m_K^2 \left(1 - \frac{1}{\gamma_K^2}\right);$$

$$\gamma_J^2 m_J^2 - m_J^2 = \gamma_K^2 m_K^2 - m_K^2;$$ ← this is the consequence of the law of conservation of momentum.

②

Using the law of conservation of energy we can eliminate $\gamma_J^2 m_J^2$:

$$\gamma_J m_J = m_B - \gamma_K m_K;$$

therefore, we have

$$(m_B - \gamma_K m_K)^2 - m_J^2 = \gamma_K^2 m_K^2 - m_K^2;$$

$$m_B^2 - 2\gamma_K m_K m_B + \cancel{\gamma_K^2 m_K^2} - m_J^2 = \cancel{\gamma_K^2 m_K^2} - m_K^2;$$

cancel out

$$2\gamma_K m_K m_B = m_B^2 - m_J^2 + m_K^2;$$

$$\boxed{\gamma_K = \frac{1}{2} \cdot \frac{m_B^2 - m_J^2 + m_K^2}{m_K m_B}}$$

$$p_K = \gamma_K m_K u_K = \gamma_K m_K \sqrt{1 - \frac{1}{\gamma_K^2}} = \boxed{m_K \sqrt{\gamma_K^2 - 1}}$$

Substituting numerical parameters, we have:

$$\begin{cases} \gamma_K = 3.5256 \\ p_K = 1682.5 \frac{\text{MeV}}{c} \end{cases}$$

(3)

Part (b). Decay of K-meson $K \to \pi^+ + \pi^-$.

In the rest frame of K:

$$m_K c^2 = E_{\pi^+} + E_{\pi^-} = \sqrt{p_{\pi^+}^2 c^2 + m_{\pi^+}^2 c^4} + \sqrt{p_{\pi^-}^2 c^2 + m_{\pi^-}^2 c^4}$$

Due to momentum conservation $p_{\pi^+} = p_{\pi^-}$

$$\Rightarrow m_K c^2 = 2\sqrt{p_\pi^2 c^2 + m_\pi^2 c^4}.$$

Putting $c = 1 \to m_K^2 = 4p_\pi^2 + 4m_\pi^2$;

$$p_\pi^2 = \frac{m_K - 4m_\pi^2}{4} \left(= \frac{m_K^2 c^4 - 4m_\pi^2 c^4}{4c^2} \right);$$

$$\boxed{p_\pi = \frac{1}{2} \sqrt{m_K^2 - 4m_\pi^2}};$$

Substituting numerical parameters,

$$\boxed{p_\pi = 206.01\ MeV/c}$$

(4)

→ Alternative solution for part (b)

The same equations as we deduced in part "a", are valid in K^0 rest-frame.

$$\gamma_\pi = \frac{1}{2} \cdot \frac{m_K^2 - m_\pi^2 + m_\pi^2}{m_K \cdot m_\pi} = \frac{1}{2} \cdot \frac{m_K}{m_\pi} = \frac{1}{2} \cdot \frac{497.67}{139.57}$$

$$\Rightarrow \boxed{\gamma_\pi = 1.783}$$

and $\boxed{p_\pi = \left(\sqrt{\gamma_\pi^2 - 1}\right) \cdot m_\pi} = 206.01\ MeV/c$

(c) Maximal opening angle

We know that, in the kaon rest frame, the two pions must emerge back to back with spatial momenta equal in magnitude but opposite in direction because the spatial momentum vanishes in that frame. Since they have the same mass, their velocities will thus also be equal in magnitude and opposite in direction, and their energies must be equal. If we let θ' be the angle that the π^+ velocity makes with the direction of the lab frame kaon momentum, then we have for the components of the 3-velocity:

$$u'_{\pi+,||} = u'_\pi \cos\theta' \qquad\qquad u'_{\pi-,||} = -\,u'_\pi \cos\theta'$$
$$u'_{\pi+,\perp} = u'_\pi \sin\theta' \qquad\qquad u'_{\pi-,\perp} = -u'_\pi \sin\theta'$$

where u'_π is the common magnitude of the pion velocity. Now, what is the opening angle in the lab frame? We must do velocity addition. Let V_K be the kaon lab frame velocity and γ_K the corresponding Lorentz factor. The pion velocities in the lab frame are

$$u_{\pi+,||} = \frac{V_K + u'_{\pi+,||}}{1 + V_K\, u'_{\pi+,||}} = \frac{V_K + u'_\pi \cos\theta'}{1 + V_K\, u'_\pi \cos\theta'} \qquad\qquad u_{\pi+,\perp} = \frac{u_{\pi+}}{\gamma_K} = \frac{u'_\pi}{\gamma_K}\sin\theta'$$
$$u_{\pi-,||} = \frac{V_K + u'_{\pi-,||}}{1 + V_K\, u'_{\pi-,||}} = \frac{V_K - u'_\pi \cos\theta'}{1 - V_K\, u'_\pi \cos\theta'} \qquad\qquad u_{\pi-,\perp} = \frac{u_{\pi-}}{\gamma_K} = -\frac{u'_\pi}{\gamma_K}\sin\theta'$$

We see that the two pions have equal and opposite perpendicular components of velocity but possibly different parallel components and so their opening angles will be different. They are

$$\tan\theta_+ = \frac{u_{\pi+,\perp}}{u_{\pi+,||}} = \frac{1 + V_K\, u'_\pi \cos\theta'}{V_K + u'_\pi \cos\theta'}\frac{u'_\pi}{\gamma_K}\,\sin\theta'$$
$$\tan\theta_- = \frac{u_{\pi+,\perp}}{u_{\pi+,||}} = \frac{1 - V_K\, u'_\pi \cos\theta'}{V_K - u'_\pi \cos\theta'}\frac{u'_\pi}{\gamma_K}\,\sin\theta'$$

(θ_+ is taken to be positive in the CCW direction, θ_- in the CW direction.) We don't need to let θ' exceed $\pi/2$ because then the π^+ and π^- just exchange places. Now, the only free parameter in the above is θ', and tan is monotonic in its argument between 0 and $\frac{\pi}{2}$, so we could in principle maximize the two angles by differentiating both expressions with respect to θ'. But we don't need to do that – we can see the result easily. Since $\tan\theta_+ = u_{\pi+,\perp}/u_{\pi+,||}$, we can maximize θ_+ by maximizing $u_{\pi+,\perp}$ and minimizing $u_{\pi+,||}$. With $\theta' \le \pi/2$, this is done by simply making $u'_{+,||} = 0$: then $u_{+,||}$ takes on its minimum value, V_K. This occurs at $\theta' = \pi/2$. We then have

$$\tan\theta_+ = \frac{1}{\gamma_K}\frac{u'_\pi}{V_K} = \tan\theta_-$$

Numerically, we have $\gamma'_\pi = 1.783$ and $\gamma_K = 3.5256$ from the previous pages (γ'_π was stated without the $'$ there), so $u'_\pi = 0.8279$ and $V_K = 0.9589$. We thus obtain $\tan\theta_\pm = 0.2449$ or $\theta_\pm = 13.76°$. So the opening angle of the vee is $\theta_+ + \theta_- = 27.5°$.

Problem 5

For this problem, since one assumes the speed of light is fixed relative to the ether, it is simplest to calculate all the travel times in the ether frame. That is, we find the positions of the various emission, absorption, and reflection events in the ether frame and use the fixed speed of light in that frame to calculate the times.

(a) In the "ether" frame,

the time t_{m1} for the trip from the mirror to M_1 is given by

$$ct_{m1} = l_1 - vt_{m1}$$

where we subtract vt_{m1} from l_1 because M_1 moves up by this amount during the travel time (because the interferometer is moving in the ether frame). So we obtain

$$t_{m1} = \frac{1}{c}\frac{l_1}{1+\beta}$$

Similary, the time t_{1m} for the trip from M_1 to the mirror is given by

$$ct_{1m} = (v(t_{m1} + t_{1m}) + l_1) - (vt_{m1})$$

where $v(t_{m1} + t_{1m}) + l_1$ is the position of the half-silvered mirror (it is at l_1 at $t = 0$ but we must account for its motion) and vt_{m1} is the position of M_1 at the instant that the light reflects off M_1 to go back to the half-silvered mirror. The vt_{m1} terms cancel, giving

$$t_{1m} = \frac{1}{c}\frac{l_1}{1-\beta}$$

t_{m2} for the trip from the mirror to M_2 is given by

$$ct_{m2} = \sqrt{l_2^2 + (vt_{m2})^2}$$

where the path length on the right side is the hypotenuse of a triangle whose sides are l_2, the horizontal distance the light beam must travel, and vt_{m2}, the vertical distance the beam must travel due to the motion of M_2. Solving for t_{m2}:

$$t_{m2} = \frac{l_2}{c\sqrt{1-\beta^2}}$$

The time t_{2m} for the trip from M_2 to the mirror is given by the hypotenuse of the triangle formed from the horizontal distance l_2 again and the vertical distance given by the distance that the half-silvered mirror moves during the travel time t_{2m}:

$$ct_{2m} = \sqrt{l_2^2 + (vt_{2m})^2}$$

$$t_{2m} = \frac{l_2}{c\sqrt{1-\beta^2}}$$

So we have

$$t_1 = t_{m1} + t_{1m} = \frac{2l_1}{c}\frac{1}{1-\beta^2}$$

$$t_2 = t_{m2} + t_{2m} = \frac{2l_2}{c}\frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta t = \frac{2l_1}{c}\frac{1}{1-\beta^2} - \frac{2l_2}{c}\frac{1}{\sqrt{1-\beta^2}}$$

After the apparatus has been rotated 90°, one has

$$\Delta t' = \frac{2l_1}{c} \frac{1}{\sqrt{1-\beta^2}} - \frac{2l_2}{c} \frac{1}{1-\beta^2}$$

So

$$\begin{aligned} c\left(\Delta t' - \Delta t\right) &= 2\left(l_1 + l_2\right)\left(\frac{1}{\sqrt{1-\beta^2}} - \frac{1}{1-\beta^2}\right) \\ &\approx 2\left(l_1 + l_2\right)\left(1 + \frac{\beta^2}{2} - 1 - \beta^2\right) \\ &= -\left(l_1 + l_2\right)\beta^2 \end{aligned}$$

(c) With the Lorentz-Fitzgerald hypothesis, t_2 won't change because the length l_2 is perpendicular to the motion and the distances the mirrors move vt_{m2} and vt_{2m} are measured in the ether rest frame. Similarly, vt_{m1} and vt_{1m} don't change, but l_1 appears contracted according to L-F, giving

$$t_1 = \frac{2l_1^0}{c} \frac{\sqrt{1-\beta^2}}{1-\beta^2} = \frac{2l_1^0}{c} \frac{1}{\sqrt{1-\beta^2}}$$

So

$$\begin{aligned} \Delta t &= \left(\frac{2l_1^0}{c} - \frac{2l_2^0}{c}\right)\frac{1}{\sqrt{1-\beta^2}} \\ \Delta t' &= \left(\frac{2l_1^0}{c} - \frac{2l_2^0}{c}\right)\frac{1}{\sqrt{1-\beta^2}} = \Delta t \end{aligned}$$

(d) We see from part (c) that

$$c\Delta t = 2\left(l_1^0 - l_2^0\right)\frac{1}{\sqrt{1-\beta^2}}$$

Problem 6

$$\begin{cases} E_i = -\partial_i \varphi - \frac{1}{c} \dot{A}_i \\ B_i = \varepsilon_{ijk} \partial_j A_k. \end{cases}$$

Then, $E^2 = (\partial_i \varphi + \frac{1}{c} \dot{A}_i)(\partial_i \varphi + \frac{1}{c} \dot{A}_i)$

$$= \partial_i \varphi \partial_i \varphi + \frac{2}{c} \dot{A}_i \partial_i \varphi + \frac{1}{c^2} (\dot{A}_i)^2.$$

$$B^2 = (\varepsilon_{ijk} \partial_j A_k) \cdot (\varepsilon_{ilm} \partial_l A_m)$$

$$= (\varepsilon_{jki} \cdot \varepsilon_{ilm}) \cdot (\partial_j A_k \cdot \partial_l A_m)$$

$$= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl})(\partial_j A_k \cdot \partial_l A_m)$$

$$= \partial_j A_k \cdot \partial_j A_k - \partial_j A_k \cdot \partial_k A_j$$

Therefore, Lagrangian density is:

$$\mathcal{L} = \frac{1}{8\pi}(E^2 - B^2) - \rho \varphi + \frac{1}{c} j_i \cdot A_i$$

$$= \frac{1}{8\pi}\left(\partial_i \varphi \cdot \partial_i \varphi + \frac{2}{c} \dot{A}_i \cdot \partial_i \varphi + \frac{1}{c^2}(\dot{A}_i)^2\right)$$

$$- \frac{1}{8\pi}\left(\partial_j A_k \cdot \partial_j A_k - \partial_j A_k \cdot \partial_k A_j\right)$$

$$- \rho \varphi + \frac{1}{c} j_i \cdot A_i$$

for φ, we have E-L equation as:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \right) + \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \varphi)} \right)$$

LHS: $\frac{\partial \mathcal{L}}{\partial \varphi} = -\rho$.

RHS: $\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi)} \right) = 0$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_i \varphi)} = \frac{1}{4\pi} \partial_i \varphi + \frac{1}{4\pi c} \dot{A}_i$$

$$\Rightarrow \quad RHS = \frac{1}{4\pi} \partial_i \partial_i \varphi + \frac{1}{4\pi c} \partial_i \dot{A}_i$$

$$\therefore \quad 4\pi \rho = -\left(\partial_i \partial_i \varphi + \frac{1}{c} \partial_i \dot{A}_i \right)$$

$$= -\partial_i \left(\partial_i \varphi + \frac{1}{c} \dot{A}_i \right)$$

note.
$$\boxed{E_i = -\partial_i \varphi - \frac{1}{c} \dot{A}_i}$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

for A_i, we have:

$$\frac{\partial \mathcal{L}}{\partial A_i} = \frac{1}{c} j_i$$

$$\frac{\partial \mathcal{L}}{\partial(\dot{A}_i)} = \frac{1}{4\pi c} \partial_i \varphi + \frac{1}{4\pi c^2} \dot{A}_i \quad \Rightarrow \quad \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\dot{A}_i)} \right) = \frac{1}{4\pi c} \partial_t \left(\partial_i \varphi + \frac{1}{c} \dot{A}_i \right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} = -\frac{1}{8\pi}\cdot\left(2\,\partial_j A_i - 2\,\partial_i A_j\right)$$

$$\therefore\ \partial_j\left(\frac{\partial \mathcal{L}}{\partial(\partial_j A_i)}\right) = -\frac{1}{4\pi}\cdot\left(\partial_j\partial_j A_i - \partial_j\partial_i A_j\right).$$

$$\therefore\ \frac{\partial \mathcal{L}}{\partial A_i} = \partial_t\left(\frac{\partial \mathcal{L}}{\partial(\dot{A}_i)}\right) + \partial_j\left(\frac{\partial \mathcal{L}}{\partial(\partial_j A_i)}\right)$$

$$\Rightarrow\ \frac{1}{c}j_i = \frac{1}{4\pi c}\,\partial_t\,\underbrace{\left(\partial_i\varphi + \frac{1}{c}\dot{A}_i\right)}_{-E_i} + \frac{1}{4\pi}\left(\partial_j\partial_i A_j - \partial_j\partial_j A_i\right).$$

now note

$$\begin{aligned}
\vec{\nabla}\times\vec{B} &= \varepsilon_{ilm}\,\partial_l B_m \\
&= \varepsilon_{ilm}\,\partial_l\cdot\left(\varepsilon_{mjk}\,\partial_j A_k\right) \\
&= \left(\delta_{ij}\,\delta_{lk} - \delta_{ik}\,\delta_{jl}\right)\left(\partial_l\partial_j A_k\right) \\
&= \underline{\partial_k\partial_j A_k - \partial_j\partial_j A_k}.
\end{aligned}$$

$$\therefore\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

Problem 7

To be provided shortly.