Physics 106a/196a – Problem Set 2 – Due Oct 13, 2006
Solutions

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Problem 1

By symmetry, the center of the mass of the wire must be at $x\hat{x}$. The mass contribution of an infinitesimal arc $d\phi$ is

$$dm = \frac{m}{a\theta} a d\phi = \frac{m}{\theta} d\phi \tag{1}$$

So we have

$$\begin{aligned} x &= \frac{1}{m}\int x dm \\ &= \frac{1}{m}\int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} a\cos\phi \frac{m}{\theta} d\phi \\ &= \frac{a}{\theta}\sin\phi \,|_{-\frac{\theta}{2}}^{\frac{\theta}{2}} \\ &= \frac{2a}{\theta}\sin\frac{\theta}{2} \\ &= a\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}} \end{aligned} \tag{2}$$

where we see when $\theta \to 0$, we get $x = a$ as expected.

Problem 2

We use the formula on Page 41 of the lecture notes

$$v = v_0 - gt + u\log\frac{m_0}{m} \tag{3}$$

In this problem, the craft hovers over the moon's surface and thereby the initial speed v_0 and final speed v are both zero. So we have

$$\begin{aligned} t &= \frac{u}{g_{Moon}}\log\frac{m_0}{m} \\ &= \frac{1500m/s}{\frac{9.8}{6}m/s^2}\log\frac{3}{2} \\ &= 372.4s \end{aligned} \tag{4}$$

Let us also derive the solution from scratch to show that it's not just a matter of finding the right equation from the lecture notes (a habit that should be discouraged because it is too easy to forget to check that the assumptions made in deriving something in lecture are the same as in the problem you are working on). We simply start with Newton's second law:

$$\frac{dp}{dt} = -m\, g_{Moon} \tag{5}$$

$$p(t+dt) - p(t) = -m\, g_{Moon} \tag{6}$$

$$(m + \dot{m}\, dt)\,(v + dv) + (-\dot{m})\,(v - u) - m\, v = -m\, g_{Moon}\, dt \tag{7}$$

As usual, multiply out and keeping only first order terms:

$$m\, dv + \dot{m}\, u\, dt = -m\, g_{Moon}\, dt \tag{8}$$

Now, set $v = 0$ and $dv = 0$ because we know v is constant and zero throughout the problem. So we have

$$\dot{m} u\, dt = -m\, g_{Moon}\, dt \tag{9}$$

$$-\frac{u}{g_{Moon}} \frac{dm}{m} = dt \tag{10}$$

$$\tag{11}$$

which can be integrated to obtain

$$\frac{u}{g_{Moon}} \log \frac{m_0}{m} = t \tag{12}$$

$$\tag{13}$$

which is the same as Eqn 4 above.

Problem 3

We choose the vertical coordinate of z wherein the origin is at the pivot and $+z$ is upward.

- Before scoop picks up the sand: Energy is conserved during this period. Initially, the potential energy of the scoop is

  $$U_0 = -m_1 g l \cos \frac{\pi}{4} \tag{14}$$

  And the potential energy of the scoop just before picking up the sand is

  $$U_1 = -m_1 g l \tag{15}$$

  Conservation of energy gives us

  $$U_0 = U_1 + \frac{1}{2} m_1 v_0^2 \tag{16}$$

  $$-m_1 g l \cos \frac{\pi}{4} = -m_1 g l + \frac{1}{2} m_1 v_0^2 \tag{17}$$

  $$v_0 = \sqrt{2gl\left(1 - \frac{\sqrt{2}}{2}\right)} \tag{18}$$

  where v_0 is the scoop's speed just before picking up the sand.

- At the moment of picking up the sand: At this time, the angular momentum of the system of the arm, the scoop and the sand relative to the pivot is conserved because the only external force except the gravity acting on the system exerts at the pivot and therefore the corresponding external torque vanishes. As for the gravity, its torque is zero since the position vector pointing from the pivot to the scoop and the sand is parallel to the vertical. *Note that you do not need to include some sort of force acting between the sand and scoop at the instant of contact – we showed in lecture that the total system linear momentum and angular momentum are affected only by external forces and torques. One has to be careful not to overthink the problem.* Before the picking up, the magnitude of the angular momentum of the system is

$$\begin{aligned}\left|\overrightarrow{L}_i\right| &= \left|\overrightarrow{L_i^{scoop}} + \overrightarrow{L_i^{arm}} + \overrightarrow{L_i^{sand}}\right| \qquad (19)\\ &= \left|\overrightarrow{L_i^{scoop}}\right| \\ &= m_1 l v_0\end{aligned}$$

where $\overrightarrow{L_i^{arm}} = 0$ since the arm has negligible weight and $\overrightarrow{L_i^{sand}} = 0$ since the initial velocity of the sand is zero. After the picking up the sand, the magnitude of the angular momentum of the system is

$$\begin{aligned}\left|\overrightarrow{L}_f\right| &= \left|\overrightarrow{L_f^{scoop}} + \overrightarrow{L_f^{arm}} + \overrightarrow{L_f^{sand}}\right| \qquad (20)\\ &= \left|\overrightarrow{L_f^{scoop}} + \overrightarrow{L_f^{sand}}\right| \\ &= m_1 l v_1 + m_2 l v_1\end{aligned}$$

where $\overrightarrow{L_f^{arm}} = 0$. Conservation of angular momentum gives us

$$\left|\overrightarrow{L}_i\right| = \left|\overrightarrow{L}_f\right| \qquad (21)$$

$$m_1 l v_0 = m_1 l v_1 + m_2 l v_1 \qquad (22)$$

$$v_1 = \frac{m_1}{m_1 + m_2} v_0 \qquad (23)$$

One could have done the above with linear momentum and forces instead of angular momentum and torques, similar equations would have appeared without the l factors.

Another way of seeing that it is momentum, and not energy, that is conserved in the collision of the scoop with the sand is to consider both energy and momentum conservation, making use of the fact that the velocity of the sand and scoop are the same after the collision:

$$m_1 v_0 = (m_1 + m_2) v_1 \qquad (24)$$

$$\frac{m_1}{2} v_0^2 = \frac{m_1 + m_2}{2} v_1^2 \qquad (25)$$

which imply

$$\frac{v_1}{v_0} = \frac{m_1}{m_1 + m_2} \qquad \frac{v_1}{v_0} = \sqrt{\frac{m_1}{m_1 + m_2}} \qquad (26)$$

Clearly, only one of the two equations can be true. You know it's possible to dissipate heat in such a collision and thereby lose mechanical energy, so one has to discard the conservation of energy equation. Note that this proof is general – any time you have a moving object hit and stick to an item at rest and the two move off together, the collision *must* be inelastic.

- After scoop picks up the sand: Conservation of energy applies here. Just after picking up the sand, the energy of the system is

$$E_2 = -(m_1 + m_2)gl + \frac{1}{2}(m_1 + m_2)v_1^2 \tag{27}$$

Assume the arm of the scoop rises to the angle θ with the vertical, where the velocity of the scoop and the sand is zero and the total energy of the system is

$$E_3 = -(m_1 + m_2)gl\cos\theta \tag{28}$$

So conservation of energy gives us

$$E_3 = E_2 \tag{29}$$

$$-(m_1 + m_2)gl + \frac{1}{2}(m_1 + m_2)v_1^2 = -(m_1 + m_2)gl\cos\theta \tag{30}$$

$$\cos\theta = 1 - \frac{1}{2}\frac{v_1^2}{gl} \tag{31}$$

$$\cos\theta = 1 - \left(\frac{m_1}{m_1 + m_2}\right)^2 \left(1 - \frac{\sqrt{2}}{2}\right) \tag{32}$$

We also notice our final result can be written as

$$\frac{1 - \cos\theta}{1 - \cos\theta_0} = \left(\frac{m_1}{m_1 + m_2}\right)^2 \tag{33}$$

where θ_0 is the angle at which the scoop is initially lifted, namely $\frac{\pi}{4}$. When m_2 is zero, $\theta = \theta_0$ since no energy is lost at the pick up.

Problem 4

We choose the vertical coordinate of z wherein the origin is at the top of the cylinder and $+z$ is upward. Let α be a parameter that describes the position along the rope, $0 \leq \alpha \leq b$, where b is the length of the rope, $\alpha = 0$ at the end that is attached to the top of the cylinder. The z position of the rope as a function of the the parameter α is

$$z(\alpha, \phi) = \begin{cases} -R(1 - \cos\frac{\alpha}{R}) & 0 \leq \alpha \leq R\theta \\ -R(1 - \cos\theta) - (\alpha - R\theta)\sin\theta & R\theta \leq \alpha \leq b \end{cases} \tag{34}$$

where $\frac{\alpha}{R}$ gives the angle of the position α with the vertical before the cutoff point between the two forms where the rope begins to unwind off the cylinder and which occurs for α such that the angle $\frac{\alpha}{R}$ of the position is θ by geometry. So the potential energy is

$$\begin{aligned} U(\theta) = &\int_{R\theta}^{b} d\alpha\lambda g\left[-R(1 - \cos\theta) - (\alpha - R\theta)\sin\theta\ \right] + \int_{0}^{R\theta} d\alpha\lambda g\left[-R(1 - \cos\frac{\alpha}{R})\right] \\ &- mg\left[R(1 - \cos\theta) + (b - R\theta)\sin\theta\right] \\ &= \lambda g\left[-R(1 - \cos\theta)(b - R\theta) - \left(\frac{b^2 - (R\theta)^2}{2} - R\theta(b - R\theta)\right)\sin\theta\ \right] \\ &+ \lambda g\left[-R(R\theta - R\sin\theta)\right] - mg\left[R(1 - \cos\theta) + (b - R\theta)\sin\theta\right] \\ &= -\lambda g(b - R\theta)\left[R(1 - \cos\theta) + \frac{b - R\theta}{2}\sin\theta\ \right] - \lambda g R^2(\theta - \sin\theta) \\ &- mg\left[R(1 - \cos\theta) + (b - R\theta)\sin\theta\right] \end{aligned} \tag{35}$$

The kinetic energy is the sum of the kinetic energies of the rotating rope and the particle

$$K(\theta) = \int_{R\theta}^{b} d\alpha \frac{1}{2}\lambda \left[(\alpha - R\theta)\dot{\theta}\right]^2 + \frac{1}{2}m\left(b - R\theta\right)^2 \dot{\theta}^2 \qquad (36)$$
$$= \frac{1}{6}\lambda \left(b - R\theta\right)^3 \dot{\theta}^2 + \frac{1}{2}m\left(b - R\theta\right)^2 \dot{\theta}^2$$

The total kinetic and potential energy of the system as a function of θ and $\dot{\theta}$ is

$$U(\theta) + K(\theta) \qquad (37)$$
$$= -\lambda g(b - R\theta)\left[R(1 - \cos\theta) + \frac{b - R\theta}{2}\sin\theta\right] - \lambda g R^2(\theta - \sin\theta)$$
$$- mg\left[R(1 - \cos\theta) + (b - R\theta)\sin\theta\right] + \frac{1}{6}\lambda\left(b - R\theta\right)^3 \dot{\theta}^2 + \frac{1}{2}m\left(b - R\theta\right)^2 \dot{\theta}^2$$

Problem 5

- Since the magnetic field is perpendicular to the plane in which a charged particle of mass m travels, it will travel along a circle with radius R. So we have

$$qvB = \frac{mv^2}{R} \qquad (38)$$
$$\Longrightarrow p = mv = \frac{qB}{1/R} \qquad (39)$$

  where v and q are the velocity and the electric charge of the charged particle, respectively, and $1/R$ is the track curvature. Eq. (39) relates the momentum of the charged particle to the track curvature directly. In order to get the velocity of the charged particle, we need know its mass m.

- Some notation:

$$m_1 = \text{mass of the proton}$$
$$m_2 = \text{mass of the nucleus}$$
$$\overrightarrow{p}_{1,i}, \overrightarrow{p}_{1,f} = \text{initial and final momentums of the proton in lab system}$$
$$\overrightarrow{p}_2 = \text{final momentums of the nucleus}$$
$$\psi_1 = \text{deflection angle of the proton } (\cos\psi_1 = \frac{\overrightarrow{p}_{1,i} \cdot \overrightarrow{p}_{1,f}}{p_{1,i}p_{1,f}})$$

  Conservation of linear momentum yields

$$\overrightarrow{p}_{1,i} = \overrightarrow{p}_{1,f} + \overrightarrow{p}_2 \qquad (40)$$
$$\overrightarrow{p}_2 = \overrightarrow{p}_{1,i} - \overrightarrow{p}_{1,f} \qquad (41)$$
$$p_2^2 = p_{1,i}^2 + p_{1,f}^2 - 2\overrightarrow{p}_{1,i} \cdot \overrightarrow{p}_{1,f} \qquad (42)$$
$$p_2^2 = p_{1,i}^2 + p_{1,f}^2 - 2p_{1,i}p_{1,f}\cos\psi_1 \qquad (43)$$

  While conservation of kinetic energy gives us

$$\frac{p_{1,i}^2}{2m_1} = \frac{p_2^2}{2m_2} + \frac{p_{1,f}^2}{2m_1} \qquad (44)$$
$$m_2 = m_1 \frac{p_2^2}{p_{1,i}^2 - p_{1,f}^2} \qquad (45)$$

Plugging Eq. (43) into the above equation yields

$$
\begin{aligned}
m_2 &= m_1 \frac{p_{1,i}^2 + p_{1,f}^2 - 2p_{1,i}p_{1,f}\cos\psi_1}{p_{1,i}^2 - p_{1,f}^2} \qquad (46)\\
&= m_1 \frac{1 + \frac{p_{1,f}^2}{p_{1,i}^2} - 2\frac{p_{1,i}p_{1,f}}{p_{1,i}^2}\cos\psi_1}{1 - \frac{p_{1,f}^2}{p_{1,i}^2}}\\
&= m_1 \frac{1 + \alpha^2 - 2\alpha\cos\psi_1}{1-\alpha^2}
\end{aligned}
$$

- Alternative solution: We start with the equation from the lecture notes

$$
\frac{T_1}{T_0} = \frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi_1 + \sqrt{\frac{m_2^2}{m_1^2} - \sin\psi_1}\right]^2 \tag{47}
$$

where we pick up the plus sign since $m_1 < m_2$ (must notice that the problem says m_2 is a nucleus, and all nuclei are heavier than a proton). We then have

$$
\begin{aligned}
\frac{(m_1+m_2)^2}{m_1^2}\frac{T_1}{T_0} &= \left[\cos\psi_1 + \sqrt{\frac{m_2^2}{m_1^2} - \sin\psi_1}\right]^2 &(48)\\
\frac{m_1+m_2}{m_1}\sqrt{\frac{T_1}{T_0}} &= \cos\psi_1 + \sqrt{\frac{m_2^2}{m_1^2} - \sin\psi_1} &(49)\\
(1+r)\,\alpha &= \cos\psi_1 + \sqrt{r^2 - \sin\psi_1} &(50)\\
r^2 - \sin\psi_1 &= \left((1+r)\,\alpha - \cos\psi_1\right)^2 &(51)
\end{aligned}
$$

$$
r^2\left(1-\alpha^2\right) + 2r\alpha\left(\cos\psi_1 - \alpha\right) + \left(-\alpha^2 + 2\alpha\cos\psi_1 - 1\right) = 0 \tag{52}
$$

where $r \equiv \frac{m_2}{m1}$ and $\sqrt{\frac{T_1}{T_0}} = \alpha$. Solving Eq. (52) yields

$$
\begin{aligned}
r &= \frac{\alpha\left(\alpha - \cos\psi_1\right) \pm \left|\alpha\cos\psi_1 - 1\right|}{1-\alpha^2} \qquad (53)\\
&= \frac{\alpha\left(\alpha - \cos\psi_1\right) \pm \left(1 - \alpha\cos\psi_1\right)}{1-\alpha^2}\\
&= \frac{\alpha^2 - 2\alpha\cos\psi_1 + 1}{1-\alpha^2} \text{ or } \frac{\alpha^2 - 1}{1-\alpha^2}
\end{aligned}
$$

where we use $|\alpha\cos\psi_1| < 1$ in the second line. Since $|\alpha| < 1$ and $r > 0$, the second solution is unphysical. So we have

$$
m_2 = m_1 \frac{1 + \alpha^2 - 2\alpha\cos\psi_1}{1-\alpha^2} \tag{54}
$$

Problem 6

We consider a system of particles in an inertial frame S_1 in which the position of particle a is $\overrightarrow{r}_a$ and the total torque on it is

$$
\overrightarrow{N} = \sum_a \overrightarrow{r}_a \times \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a}\overrightarrow{f}_{ab}\right] \tag{55}
$$

Assume there is another inertial frame S_2 travelling with the velocity $\overrightarrow{v}$ relative to S_1. So in S_2 the position of particle a is

$$\overrightarrow{r}'_a = \overrightarrow{r}_a + \overrightarrow{v}t + \overrightarrow{R} \tag{56}$$

where $\overrightarrow{R}$ is a constant vector. And the total torque in S_2 is

$$\begin{aligned}
\overrightarrow{N}' &= \sum_a \overrightarrow{r}'_a \times \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a} \overrightarrow{f}_{ab}\right] \\
&= \sum_a \left(\overrightarrow{r}_a + \overrightarrow{v}t + \overrightarrow{R}\right) \times \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a} \overrightarrow{f}_{ab}\right] \\
&= \sum_a \overrightarrow{r}_a \times \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a} \overrightarrow{f}_{ab}\right] + \left(\overrightarrow{v}t + \overrightarrow{R}\right) \times \sum_a \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a} \overrightarrow{f}_{ab}\right] \\
&= \sum_a \overrightarrow{r}_a \times \left[\overrightarrow{F}_a^{(e)} + \sum_{b\neq a} \overrightarrow{f}_{ab}\right] = \overrightarrow{N}
\end{aligned} \tag{57}$$

where $\sum_a \overrightarrow{F}_a^{(e)} = 0$ since the total force is zero and $\sum_{a,b,b\neq a} \overrightarrow{f}_{ab} = 0$ due to the weak form of Newton's second law. So the total torque is independent of which inertial frame it is calculated in.

Problem 7

- Since the scattering is isotropic in the center-of-mass frame, the probability of the particles entering a solid angle element $d\Omega$ is proportional to $\frac{d\Omega}{4\pi}$,*i.e.* equal to $d\Omega$. The distribution with respect the angle θ is obtained by putting $d\Omega = 2\pi \sin\theta d\theta$,*i.e.* the corresponding probability is

  $$dP = P(\theta)d\theta = \frac{1}{2}\sin\theta d\theta = -\frac{1}{2}d\cos\theta \tag{58}$$

  From the lecture notes, we have

  $$\frac{T_1}{T_0} = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \tag{59}$$

  $$\frac{T_2}{T_0} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \tag{60}$$

  which yields

  $$\frac{dT_1}{T_0} = d\left(\frac{T_1}{T_0}\right) = \frac{2m_1m_2}{(m_1+m_2)^2}(d\cos\theta) = -\frac{4m_1m_2}{(m_1+m_2)^2}dP \tag{61}$$

  $$\frac{dT_2}{T_0} = d\left(\frac{T_2}{T_0}\right) = \frac{2m_1m_2}{(m_1+m_2)^2}(-d\cos\theta) = -\frac{4m_1m_2}{(m_1+m_2)^2}dP \tag{62}$$

  The distribution $P_i(i=1,2)$ of $T_i(i=1,2)$ is $P_i = \left|\frac{dP}{dT_i}\right|$ so we get

  $$\begin{aligned}
  P_1 &= \frac{1}{2m_1v_1'V} \\
  &= \frac{(m_1+m_2)^2}{4m_2m_1T_0}
  \end{aligned} \tag{63}$$

$$
\begin{aligned}
P_2 &= \frac{1}{2m_2 v_2' V} \\
&= \frac{(m_1+m_2)^2}{4m_1 m_2 T_0}
\end{aligned} \tag{64}
$$

- Alternative solution: Squaring the equations

$$
\overrightarrow{v}_i = \overrightarrow{v}'_i + \overrightarrow{V} \qquad i = 1, 2 \tag{65}
$$

where $\overrightarrow{v}_i$ is final velocity of particle i in lab system, $\overrightarrow{v}'_i$ is final velocity of particle i in cm system and $\overrightarrow{V}$ is velocity of cm system with respect to lab system, we have

$$
v_i^2 = v_i'^2 + V^2 + 2v_i' V \cos\theta \tag{66}
$$

Since $\left|\overrightarrow{v}'_i\right|$ and $\left|\overrightarrow{V}\right|$ are constant, we have

$$
d\left(v_i^2\right) = 2v_i' V d\left(\cos\theta\right) \tag{67}
$$

$$
d\left(\cos\theta\right) = \frac{d\left(v_i^2\right)}{2v_i' V} = \frac{dT_i}{m_i v_i' V} \tag{68}
$$

where $T_i = \frac{1}{2} m_i v_i^2$ is the final state kinetic energy of particle i in the lab frame. Substituting Eq. (68) in Eq. (58), the distribution P_i of T_i is

$$
P_i dT_i = \frac{dT_i}{2m_i v_i' V} \tag{69}
$$

where

$$
V = v_2' = \frac{m_1 u_1}{m_1+m_2} = \frac{m_1\sqrt{\frac{2T_0}{m_1}}}{m_1+m_2} = \frac{\sqrt{2m_1 T_0}}{m_1+m_2} \tag{70}
$$

$$
v_1' = \frac{m_2 u_1}{m_1+m_2} = \frac{m_2\sqrt{\frac{2T_0}{m_1}}}{m_1+m_2} = \frac{m_2\sqrt{2m_1 T_0}}{m_1\left(m_1+m_2\right)} \tag{71}
$$

So we have

$$
\begin{aligned}
P_1 &= \frac{1}{2m_1 v_1' V} \\
&= \frac{(m_1+m_2)^2}{4m_2 m_1 T_0}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
P_2 &= \frac{1}{2m_2 v_2' V} \\
&= \frac{(m_1+m_2)^2}{4m_1 m_2 T_0}
\end{aligned} \tag{73}
$$