Physics 106a/196a – Problem Set 5 – Due Nov 10, 2006
Solutions

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Version 2: December 3, 2006

Version 2: Correction of equation for $\dot{p}_\theta$, had extra power of r in denominator (p. 4 of soln set).

Problem 1

The Lagrangian is

$$L = \frac{1}{2}M\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{1}{2}m\left(\dot{x}_p^2 + \dot{y}_p^2\right) - mgy_p - Mgy_b$$

The constraint equations are

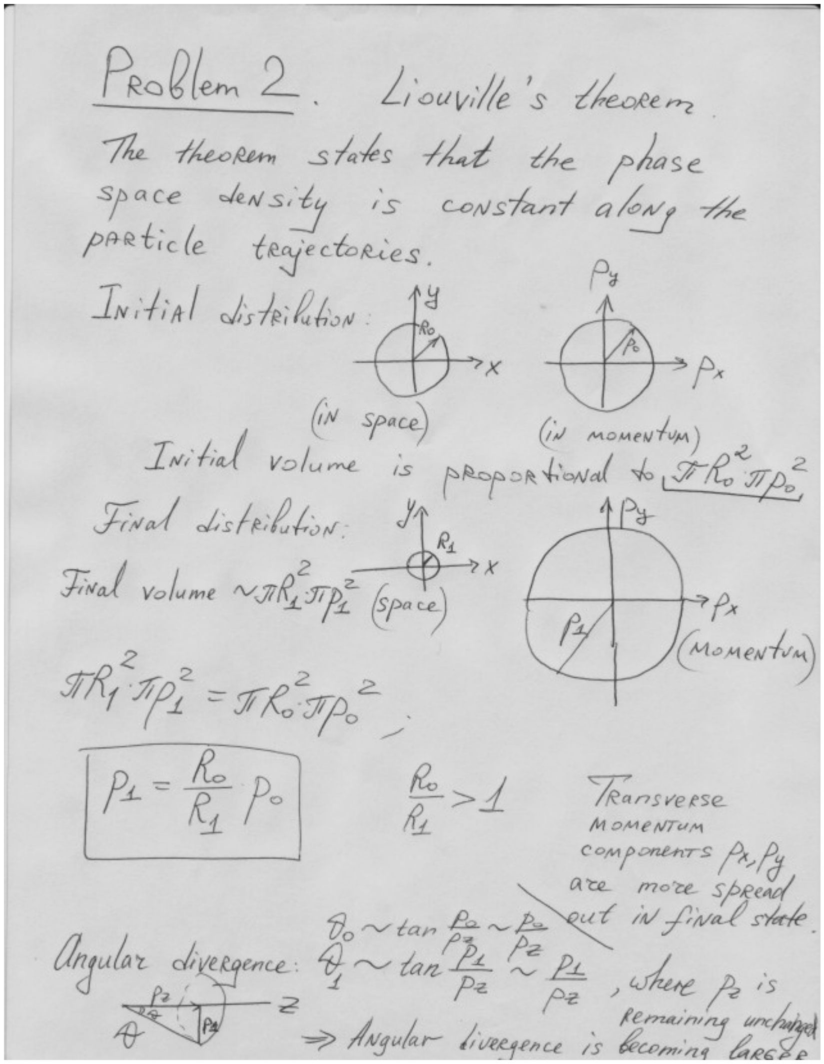
$$\begin{aligned} G_p &= \sqrt{(x_p - x_b)^2 + (y_p - y_b)^2} - l \\ G_b &= y_b \end{aligned}$$

where l is the length of the pendulum. The resulting Euler-Lagrange equations are

$$\begin{aligned} x_b &: -M\ddot{x}_b + \lambda_p \frac{x_b - x_p}{\sqrt{(x_p - x_b)^2 + (y_p - y_b)^2}} = 0 \\ y_b &: -M\ddot{y}_b - Mg + \lambda_p \frac{y_b - y_p}{\sqrt{(x_p - x_b)^2 + (y_p - y_b)^2}} = 0 \\ x_p &: -m\ddot{x}_p + \lambda_p \frac{x_p - x_b}{\sqrt{(x_p - x_b)^2 + (y_p - y_b)^2}} = 0 \\ y_p &: -m\ddot{y}_p - mg + \lambda_b \frac{y_p - y_b}{\sqrt{(x_p - x_b)^2 + (y_p - y_b)^2}} + \lambda_b = 0 \end{aligned}$$

We do not ask you to solve for the accelerations in this problem because the equations are more complicated than in the Midterm Exam Problem 4. In that case, because the constraint *and* the potential energy were linear in the coordinates, the equations of motion ended up being linear in the accelerations and the Lagrange multipliers with the coordinates not appearing. The linearity guaranteed that an algebraic solution for the accelerations and Lagrange multipliers was possible. (If you wrote the constraint in a nonlinear way, the coordinates would appear in the equations of motion, but, since those equations of motion would have to be equivalent to a set in which the coordinates do not appear, the existence of an algebraic solution remained guaranteed.) In this case, the nonlinearity of the constraint (no matter what way it is written) ruins that, giving equations of motion with both the acceleration and the coordinate. These are differential equations that require more complicated means of solution. There may still be an algebraic solution, but it is no longer guaranteed. The same difficulty would have occurred if the potential energy had been nonlinear in the coordinates.

Problem 2



Problem 2. Liouville's theorem.

The theorem states that the phase space density is constant along the particle trajectories.

Initial distribution:

y, R_0, x (in space) p_y, p_0, p_x (in momentum)

Initial volume is proportional to $\pi R_0^2 \pi p_0^2$

Final distribution:

y, R_1, x p_y, p_x, p_1

Final volume $\sim \pi R_1^2 \pi p_1^2$ (space) (momentum)

$$\pi R_1^2 \pi p_1^2 = \pi R_0^2 \pi p_0^2;$$

$$\boxed{p_1 = \frac{R_0}{R_1} p_0} \qquad \frac{R_0}{R_1} > 1$$

Transverse momentum components p_x, p_y are more spread out in final state.

$\theta_0 \sim \tan\frac{p_0}{p_z} \sim \frac{p_0}{p_z}$

Angular divergence: $\theta_1 \sim \tan\frac{p_1}{p_z} \sim \frac{p_1}{p_z}$, where p_z is remaining unchanged

p_z, z, p_1, θ

$\Rightarrow$ Angular divergence is becoming larger

Problem 3

The Virial Theorem tells us that

$$\langle T \rangle = \frac{n}{2} \langle U \rangle \tag{1}$$

for potential energies that are power laws in the coordinate, $U = k\,r^n$. For the harmonic oscillator, $n = 2$, so

$$\langle T \rangle = \langle U \rangle \tag{2}$$

The total energy is therefore

$$\langle E \rangle = \langle T + U \rangle = 2\langle T \rangle = 3\,N\,k\,\Theta \tag{3}$$

where N is the number of atoms, k is Boltzmann's constant, and Θ is the temperature. The heat capacity is then

$$C = \frac{dE}{d\Theta} = 3\,N\,k \tag{4}$$

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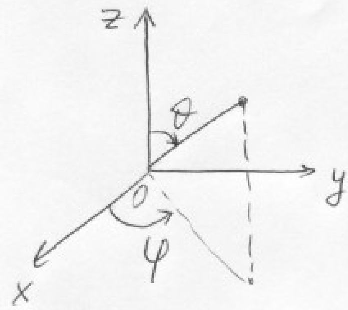
Problem 4. Motion in a spherically-symmetric field.

→ Calculation of Hamiltonian in spherical coordinates.



$$\begin{cases} x = r \sin\theta \cdot \cos\phi \\ y = r \cdot \sin\theta \cdot \sin\phi \\ z = r \cdot \cos\theta \end{cases}$$

minus

$$\begin{cases} \dot{x} = \dot{r} \sin\theta \cos\phi + r \cos\theta \cdot \cos\phi \cdot \dot{\theta} + (-1) \cdot r \sin\phi \cdot \sin\theta \, \dot{\phi} \, ; \\ \dot{y} = \dot{r} \sin\theta \sin\phi + r \cdot \sin\phi \cos\theta \cdot \dot{\theta} + r \cdot \sin\theta \cos\phi \, \dot{\phi} ; \\ \dot{z} = \dot{r} \cdot \cos\theta + (-1) \cdot r \cdot \sin\theta \cdot \dot{\theta} \end{cases}$$

Therefore, $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \ldots = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2$

The Lagrangian is

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) + \frac{k}{r} \tag{5}$$

The canonical momenta are

$$p_r = m \dot{r} \qquad p_\theta = m r^2 \dot{\theta} \qquad p_\phi = m r^2 \sin^2\theta \, \dot{\phi} \tag{6}$$

The first is the radial linear momentum. The latter two are the angular momenta in the θ and ϕ coordinates. The Hamiltonian is

$$H = \sum_k p_k \dot{q}_k - L \tag{7}$$

$$= \dot{r} \, p_r + \dot{\theta} \, p_\theta + \dot{\phi} \, p_\phi - \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) - \frac{k}{r} \tag{8}$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} + \frac{p_\phi^2}{2 m r^2 \, sin^2\theta} - \frac{k}{r} \tag{9}$$

where one uses the definitions of the canonical momenta to eliminate the generalized velocities $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ so that the Hamiltonian is written as a function of the coordinates and canonical momenta only.

Note: $\dot{p}_\theta$ on this page corrected, 2006/12/03.

Canonical equations of motion.

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A) $\dfrac{dp_j}{dt} = \dot{p}_j = -\dfrac{\partial H}{\partial q_j}$

$$\frac{\partial H}{\partial r} = \frac{1}{2m}\left(-\frac{2p_\theta^2}{r^3} - \frac{2p_\phi^2}{r^3\sin^2\theta}\right) + \frac{k}{r^2} = -\frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + \frac{k}{r^2};$$

$$\frac{\partial H}{\partial \theta} = -\frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}; \quad \frac{\partial H}{\partial \phi} = 0;$$

Equations of motion:

$$\boxed{\begin{aligned} \dot{p}_\phi &= 0; \\ \dot{p}_r &= \frac{p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}}{mr^3} - \left(\frac{k}{r^2}\right); \\ \dot{p}_\theta &= \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}; \end{aligned}}$$

B) $\dot{q}_j = \dfrac{\partial H}{\partial p_j} \Rightarrow \begin{aligned} \frac{\partial H}{\partial p_r} &= \frac{p_r}{m} \\ \frac{\partial H}{\partial p_\theta} &= \frac{p_\theta}{mr^2} \\ \frac{\partial H}{\partial p_\phi} &= p_\phi/(mr^2\sin^2\theta) \end{aligned} \Rightarrow \begin{aligned} \dot{r} &= \frac{p_r}{m}; \\ \dot{\theta} &= \frac{p_\theta}{mr^2}; \\ \dot{\phi} &= \frac{p_\phi}{mr^2\sin^2\theta} \end{aligned}$

Conservation of Angular momentum

4

By now we were solving the problem in three dimensions. It is quite instructive, but in fact it was "an attempt to shoot into sparrows from a big canon", or killing a mosquitto with a hammer. We will show now that the motion takes place in a plane, so spherical coordinates are unnecessary here, polar coordinates are enough for us.

Consider angular momentum $\vec{L} = [\vec{r} \times \vec{p}]$:

$$\frac{d}{dt}\vec{L} = \left[\frac{d\vec{r}}{dt} \times \vec{p}\right] + \left[\vec{r} \times \frac{d\vec{p}}{dt}\right] =$$

$$= \underbrace{[\vec{v} \times (m\vec{v})]}_{\text{collinear vectors}} + [\vec{r} \times \vec{F}] = 0 + 0 = 0.$$

$\vec{F} = -\text{grad}\,U = -\dfrac{k}{r^2}\cdot\hat{r}$ — again collinear vectors

From linear algebra: Equation $[\vec{r} \times \vec{p}] = \vec{L}_0 = \text{const}$ determines a plane spanned by $\vec{r}$ and $\vec{p}$, orthogonal to $\vec{L}_0$

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→ Polar coordinates, four-dimensional phase space.

It is convenient to consider the following coordinate system:

axis z is parallel to conserved momentum $\vec{L}_0$,

Angle θ is constant $= \frac{\pi}{2}$; $\dot{\theta} = 0$;

Reduce all expressions (for H, p_j, and equations of motions) to polar coordinates r, φ.

We have:

Lagrangian $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{k}{r}$;

Hamiltonian $\boxed{H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r}}$;

Equations of motion: $\dot{r} = \frac{p_r}{m}$; $\dot{\phi} = \frac{p_\phi}{mr^2}$;

$\dot{p}_r = \frac{p_\phi^2}{mr^3} - \frac{k}{r^2}$; $\dot{p}_\phi = 0$;

(We no longer consider p_θ, because formally from previous results, $\dot{\theta} = \frac{p_\theta}{mr^2} = 0 \Rightarrow p_\theta = 0$)

So, we have four-dimensional phase space: $\underbrace{r; \phi; p_r}_{\text{nontrivial}}$; $\underbrace{p_\phi}_{\text{const}}$

The problem asked to make θ the second spatial coordinate, not ϕ. But it easy to see there is no difference between the two cases. Instead of choosing the conserved angular momentum to be along the z axis, we could have chosen the plane of the motion to correspond to a constant value of ϕ, which we will call ϕ_0, placing the conserved angular momentum in the xy plane at an angle $\phi_0 + \pi/2$ from the x axis (up to a sign). Then we have

$$\phi = \phi_0 \qquad p_\phi = 0 \qquad \dot{\phi} = 0 \tag{10}$$

Inserting these in the equations of motion, we obtain

$$\dot{p}_\theta = 0 \qquad \dot{p}_r = \frac{p_\theta^2}{m\,r^3} - \frac{k}{r^2} \qquad \dot{\theta} = \frac{p_\theta}{m\,r^2} \qquad \dot{r} = \frac{p_r}{m} \tag{11}$$

Thus, we obtain the same equations of motion but with ϕ replaced by θ everywhere. The rest of the solution thus carries through.

→ Phase portrait on the $(r - p_r)$ plane.

Total energy $E_0 = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{k}{r} = \text{CONST}.$

We already know: $p_\phi = \text{const}$;

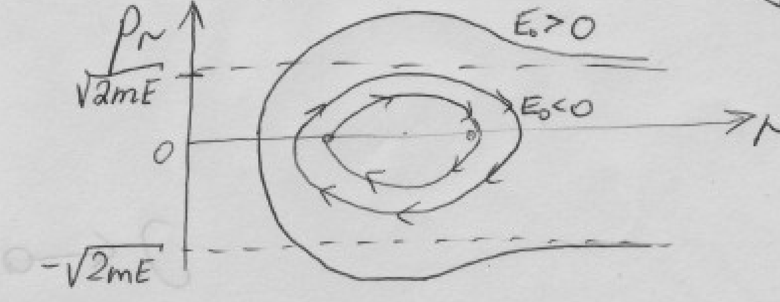
$$2mE_0 = \frac{r^2 p_r^2}{r^2} + \frac{p_\phi^2}{r^2} - \frac{2mkr}{r^2}\ ; \quad p_r^2 = -\frac{p_\phi^2}{r^2} + \frac{2mk}{r} + 2mE_0$$

$$p_r = \pm\sqrt{-\frac{p_\phi^2}{r^2} + \frac{2mk}{r} + 2mE} = \pm\frac{\sqrt{2mEr^2 + 2mkr - p_\phi^2}}{r}$$

We need to perform some mathematical analyzis of this equation, such as finding roots, investigation of max/min, convex/concave

Roots: $p_r = 0$ if $r_{1,2} = \frac{-2mk \pm \sqrt{4m^2k^2 + 8mp_\phi^2 E}}{4mE}$;

Character of $p_r - r$ picture depends strongly on the sign of total energy! 1) If $E_0 > 0$ a particle goes to infinity, 2) if $E_0 < 0$, the trajectory of particle is bounded.



Problem 5

We check Poisson Bracket

$$
\begin{aligned}
[Q, P]_{q,p} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= -2i \neq 1
\end{aligned}
$$

which implies the transformation is not canonical. We can choose (by trial and error, or by writing $P' = aq + ibp$ and then finding appropriate values of a and b to obtain the desired result)

$$
Q' = q + ip \qquad P' = \frac{i}{2}(q - ip)
$$

so that

$$
\left[Q', P'\right]_{q,p} = 1
$$

which means this transformation is canonical. Solve for p, P' in terms of q, Q' (by simple algebra):

$$
p = \frac{Q' - q}{i} \qquad P' = i\left(q - \frac{Q'}{2}\right)
$$

The generating function $F(q, Q')$ therefore must satisfy

$$
\frac{\partial F}{\partial q} = p = i\left(q - Q'\right) \qquad \frac{\partial F}{\partial Q'} = -P = -i\left(q - \frac{Q'}{2}\right)
$$

which means the generating function could be

$$
F\left(q, Q'\right) = i\left(\frac{q^2}{2} - qQ' + \frac{Q'^2}{4}\right)
$$

One can obtain the above by integrating the partial derivatives, taking care with the terms that depend on both q and Q.

Problem 6

See the following page for the solution. A small addendum: Though it was not specifically asked, we note that the fact that the Jacobian determinant of a canonical transformation is 1 not only implies that the transformation is invertible, but also that its inverse also has Jacobian determinant 1 and thus is also a canonical transformation.

(1)

Problem 5. Let the original sympletic coordinates be $\vec{x}$;

let's consider two transformed sets of coordinates $\vec{\xi}$ and $\vec{\eta}$:

$d\vec{\xi} = \mathcal{J}_\xi d\vec{x}$ and $d\vec{\eta} = \mathcal{J}_\eta d\vec{x}$.

If those two transformations are canonical,

$\frac{d\vec{x}}{dt} = \Gamma \vec{\nabla}_x H$; $\frac{d\vec{\eta}}{dt} = \Gamma \cdot \vec{\nabla}_\eta H$; $\frac{d\vec{\xi}}{dt} = \Gamma \vec{\nabla}_\xi H$;

Consider a new set of coordinates $\vec{z}$:

$d\vec{z} = \mathcal{J}_\eta \cdot d\vec{\xi} = \mathcal{J}_\eta \mathcal{J}_\xi d\vec{x}$ ($\vec{z}$ is the composition of the above two transformations)

$\frac{d\vec{z}}{dt} = \mathcal{J}_\eta \mathcal{J}_\xi \frac{d\vec{x}}{dt} = \mathcal{J}_\eta \mathcal{J}_\xi \Gamma \vec{\nabla}_x H$;

$\vec{\nabla}_x H = (\mathcal{J}_\eta \mathcal{J}_\xi)^T \vec{\nabla}_z H$ (from lectures)

$\frac{d\vec{z}}{dt} = (\mathcal{J}_\eta \mathcal{J}_\xi)\Gamma(\mathcal{J}_\eta \mathcal{J}_\xi)^T \vec{\nabla}_z H$;

We need $(\mathcal{J}_\eta \mathcal{J}_\xi)\Gamma(\mathcal{J}_\eta \mathcal{J}_\xi)^T = \Gamma$ for the composed transformation to be canonical.

(2)

It follows from the fact that η and ξ are canonical transformations that:

$\mathcal{J}_\eta \cdot \Gamma \cdot \mathcal{J}_\eta^T = \Gamma$; $\mathcal{J}_\xi \Gamma \mathcal{J}_\xi^T = \Gamma$;

$(\mathcal{J}_\eta \cdot \mathcal{J}_\xi)\Gamma(\mathcal{J}_\eta \mathcal{J}_\xi)^T = \mathcal{J}_\eta(\mathcal{J}_\xi \Gamma \mathcal{J}_\xi^T)\mathcal{J}_\eta^T$

$= \mathcal{J}_\eta \Gamma \mathcal{J}_\eta^T = \Gamma$;

Therefore, the composition, i.e. the product, of two canonical transformations is also canonical.

→ Any canonical transformation is invertible because its Jacobian determinant is 1, i.e. non-zero.

Problem 7

The potential energy is time independent so the Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{k}{|x|} \equiv E$$

and has constant value E. The particle is bound (and thus the system is periodic) when $E < 0$. Therefore, we may find $p = p(q, E)$:

$$p = \sqrt{2m\left(E + \frac{k}{|x|}\right)}$$

from which we see the particle oscillates between $\frac{k}{E}$ and $-\frac{k}{E}$. The action variable is thus

$$\begin{aligned}
I &= \frac{1}{2\pi}\oint p\,dx \\
&= \frac{1}{2\pi}2\int_{\frac{k}{E}}^{-\frac{k}{E}} \sqrt{2m\left(E + \frac{k}{|x|}\right)}\,dx \\
&= \frac{2\sqrt{2m}}{\pi}\int_0^{-\frac{k}{E}} \sqrt{E + \frac{k}{x}}\,dx \\
&= \frac{2\sqrt{2m}}{\pi}\sqrt{-E}\left(\frac{k}{-E}\right)\int_0^{-\frac{k}{E}} \sqrt{-1 + \frac{1}{\left(\frac{-Ex}{k}\right)}}\,d\left(\frac{-Ex}{k}\right) \\
&= \frac{2\sqrt{2m}}{\pi}\frac{k}{\sqrt{-E}}\int_0^1 \sqrt{-1 + \frac{1}{y}}\,dy \\
&= \frac{2\sqrt{2}k}{\pi}\sqrt{\frac{m}{-E}}\frac{\pi}{2} \\
&= k\sqrt{\frac{2m}{-E}} \\
\Rightarrow E &= -\frac{2mk^2}{I^2}
\end{aligned}$$

where $y = \frac{-Ex}{k}$. Note that, even if you were unable to do the difficult integral above, it should be clear that the integral is just a constant numerical multiplier because it no longer depends on any of the parameters of the problem. One could just write it as α and leave it undefined, and obtain the rest of the solution with α as a free parameter. A point or two would be deducted, but you would reive most of the credit for the problem. This is a good example of moving on in a problem if you get stuck on some part that is not relevant to the physics.

The oscillation period is

$$\begin{aligned}
T &= 2\pi\left(\frac{\partial E}{\partial I}\right)^{-1} \\
&= \frac{2\pi I^3}{4k^2 m} \\
&= \pi k\sqrt{\frac{2m}{-E^3}}
\end{aligned}$$

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PROBLEM 8.

We deal with SPHERICAL COORDINATES:

$$\vec{X} = \begin{pmatrix} r\cdot\sin\theta\cos\varphi \\ r\cdot\sin\theta\cdot\sin\phi \\ r\cdot\cos\theta \end{pmatrix}, \quad \dot{\vec{X}} = \dot{r}\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} + r\dot{\theta}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} + r\dot{\phi}\begin{pmatrix} -\sin\theta\cdot\sin\varphi \\ -\sin\theta\cdot\cos\varphi \\ 0 \end{pmatrix},$$

Kinetic energy:

$$T = \frac{1}{2}m\dot{\vec{x}}^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right).$$

Potential, given in the problem, is velocity-independent

Therefore canonical momenta are $p_r = \frac{\partial T}{\partial \dot{r}} = M\dot{r}$;

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = Mr^2\dot{\theta};$$

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = Mr^2\sin^2\theta\,\dot{\phi};$$

Hamiltonian $H(r,\theta,\varphi) = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r,\theta,\varphi)$

OR $H(r,\theta,\varphi) = \left[\frac{p_r^2}{2m} + V_r(r)\right] + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + V_\theta(\theta)\right] + \frac{1}{r^2\sin^2\theta}\left[\frac{p_\varphi^2}{2m} + V_\varphi(\varphi)\right]$

Introduce $S(\vec{q},\vec{\alpha},t) = W(\vec{q},\vec{\alpha}) - Et$ (PRINCIPAL FUNCTION),

HAMILTONIAN-JACOBI EQUATION is in RESTRICTED FORM:

$$H\left(\vec{q},\frac{\partial W}{\partial \vec{q}}\right) = \left[\frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + V_r(r)\right] + \frac{1}{r^2}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial \theta}\right)^2 + V_\theta(\theta)\right] +$$

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$$+ \frac{1}{r^2\sin^2\theta}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial \phi}\right)^2 + V_\phi(\phi)\right] = E;$$

All terms that depend on ϕ are just the third term above; the first two terms in the bottom of the previous page do not depend on ϕ.

Therefore, we must have $\frac{1}{2m}\left(\frac{\partial W}{\partial \phi}\right)^2 + V_\phi(\phi) = \alpha_\phi$

CONSTANT

$$\Rightarrow H\left(\vec{q},\frac{\partial W}{\partial \vec{q}}\right) = \left[\frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + V_r(r)\right] +$$

$$+ \frac{1}{r^2}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial \theta}\right)^2 + V_\theta(\theta) + \frac{\alpha_\phi}{\sin^2\theta}\right] = E;$$

IN the last expression FOR HAMILTONIAN all dependence on θ in contained in the second term only; the first term does not depend on θ.

Therefore: $\frac{1}{2m}\left(\frac{\partial W}{\partial \theta}\right)^2 + V_\theta(\theta) + \frac{\alpha_\phi}{\sin^2\theta} = \alpha_\theta = \text{CONST}$

AND Hamiltonian is: $H\left(\vec{q},\frac{\partial W}{\partial \vec{q}}\right) = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + V(r) + \frac{\alpha_\theta}{r^2} = E$

(3)

Resumé:

Hamilton – Jacobi equation is separable with

$$\begin{cases} S(\vec{q}, \vec{\alpha}, t) = W_r(r, \vec{\alpha}) + W_\theta(\theta, \vec{\alpha}) + W_\phi(\phi, \vec{\alpha}) - \\ \qquad - E \cdot t \, ; \\ \alpha_\phi = \frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + V_\phi(\phi) \, ; \\ \alpha_\theta = \frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + V_\theta(\theta) + \frac{\alpha_\phi}{\sin^2\theta} \\ E = \frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + V_r(r) + \frac{\alpha_\theta}{r^2} \end{cases}$$

(here α_ϕ, α_θ and E are constants)

→ Equations of motion:

$$p_r = \frac{\partial S}{\partial r} = \frac{\partial W_r}{\partial r} = \pm\sqrt{2m\left(E - V_r(r)\right) - \frac{\alpha_\theta}{r^2}}$$

$$p_\theta = \frac{\partial S}{\partial \theta} = \frac{\partial W_\theta}{\partial \theta} = \pm\sqrt{2m\left(\alpha_\theta - V_\theta(\theta)\right) - \frac{\alpha_\phi}{\sin^2\theta}}$$

$$p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial W_\phi}{\partial \phi} = \pm\sqrt{2m\left(\alpha_\phi - V_\phi(\phi)\right)}$$

For completeness, we note that the last set of three equations are integrable equations for the three functions $W_r(r, \vec{\alpha})$, $W_\theta(\theta, \vec{\alpha})$, and $W_\phi(\phi, \vec{\alpha})$ where $\vec{\alpha} = (E, \alpha_\theta, \alpha_\phi)$ are constants set by initial conditions. Another set of three equations for the canonical coordinates $\vec{\beta}$ are simply the partial derivatives of the W functions (Equations 2.68 and 2.69 of the lecture notes):

$$\begin{aligned} t + \beta_1' = \beta_1 &= \frac{\partial W_r(r, \vec{\alpha})}{\partial E} \\ \beta_\theta &= \frac{\partial W_\theta(\theta, \vec{\alpha})}{\partial \alpha_\theta} \\ \beta_\phi &= \frac{\partial W_\phi(\phi, \vec{\alpha})}{\partial \alpha_\phi} \end{aligned}$$

The values of $\vec{\alpha}$ and $\vec{\beta}$ are obtained from the initial conditions, and then we know that $\vec{\alpha}$, β_θ, and β_ϕ are constant and β_1 evolves linearly in time by dint of the canonical transformation generated by the function S. If one inverts to write (r, θ, ϕ) and (p_r, p_θ, p_ϕ) in terms of $\vec{\alpha}$ and $\vec{\beta}$, one thus has the full solution to the equations of motion.