Physics 106a/196a – Problem Set 6 – Due Nov 17, 2006
Solutions

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Version 3: December 4, 2006

Version 2: Corrected solution to 3(b)(iii).
Version 3: Corrected missing exponents on 4th page of solution to 3.

Problem 1

(a) The Hamiltonian is

$$H = \frac{p^2}{2m} \equiv E$$

and has constant value E. Therefore, we may find $p = p(E)$:

$$p = \sqrt{2mE}$$

from which we see the particle oscillates between 0 and L. The action variable is thus

$$\begin{aligned} I &= \frac{1}{2\pi}\oint p dx \\ &= \frac{1}{2\pi}2\int_0^L \sqrt{2mE}dx \\ &= \frac{L}{\pi}\sqrt{2mE} \\ &\Rightarrow E = \frac{\pi^2 I^2}{2mL^2} \end{aligned} \tag{1}$$

The oscillation period is

$$\begin{aligned} T &= 2\pi\left(\frac{\partial E}{\partial I}\right)^{-1} \\ &= 2\pi\frac{mL^2}{\pi^2 I} \\ &= \frac{2mL^2}{L\sqrt{2mE}} \\ &= \frac{2L}{\sqrt{\frac{2E}{m}}} \end{aligned}$$

On the other hand, the velocity of the particle with the kinetic energy E is given by

$$v = \sqrt{\frac{2E}{m}}$$

So the period of the motion is

$$T = \frac{2L}{\sqrt{\frac{2E}{m}}}$$

(b) The momentum of the particle is

$$p = \sqrt{2mE}$$

During one period, $\Delta t = \frac{2L}{\sqrt{\frac{2E}{m}}}$, we have

$$\begin{aligned} \langle F \rangle \Delta t &= 2p \\ \langle F \rangle &= \frac{2p}{\Delta t} = \frac{2\sqrt{2mE}}{\frac{2L}{\sqrt{\frac{2E}{m}}}} = \frac{2E}{L} \end{aligned}$$

Adiabatic invariance implies I is constant to first order even as L varies. So Eq. (1) implies

$$\begin{aligned} 0 = \frac{dI}{dt} &= \frac{d}{dt}\left(\frac{L}{\pi}\sqrt{2mE}\right) \\ &\Rightarrow \sqrt{E}\frac{dL}{dt} + \frac{L}{2\sqrt{E}}\frac{dE}{dt} = 0 \\ &\Rightarrow \frac{dE}{dt} = -\frac{2E}{L}\frac{dL}{dt} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \frac{d\langle F \rangle}{dt} &= \frac{d}{dt}\left(\frac{2E}{L}\right) = \frac{2}{L}\frac{dE}{dt} - \frac{2E}{L^2}\frac{dL}{dt} \\ &= -\frac{6E}{L^2}\frac{dL}{dt} = \frac{3}{L}\frac{dE}{dt} \end{aligned}$$

Eq. (2) gives us

$$\frac{dE}{dt} = -\frac{2E}{L}\frac{dL}{dt} \Rightarrow \frac{d\left(\ln E + 2\ln L\right)}{dt} = 0 \Rightarrow EL^2 = \text{constant}$$

since $V = L^3$ and $E \propto T$ for ideal gas then we have

$$TV^{\frac{2}{3}} = \text{constant}$$

which are the ideal monatomic gas adiabatic relation $TV^{\frac{2}{3}}$ =constant.

Problem 2

(a) The solution to the EOM

$$m\ddot{q} + \frac{m\omega}{Q}\dot{q} + m\omega^2 q = F_0 e^{i\phi_0} e^{i\omega' t}$$

takes the form

$$q(t) = q_p(t) + q_h(t)$$

where

$$q_p(t) = \mathrm{Re}\left[\frac{F_0 e^{i\phi_0} e^{i\omega' t}}{m\omega^2 - m\omega'^2 + \frac{im\omega\omega'}{Q}}\right]$$

is the steady-state term and

$$\begin{aligned}
q_h(t) &= \exp\left(-\frac{\omega t}{2Q}\right)\left[A_1 \cos\omega t + A_2 \sin\omega t\right] \\
&= \mathrm{Re}\left[A \exp\left(-\frac{\omega t}{2Q}\right)\exp\left(i\omega t + i\phi\right)\right]
\end{aligned}$$

is the transient term. If there is no transient term, on has $A_1 = A_2 = 0$ and then obtains

$$\begin{aligned}
q(t) &= \mathrm{Re}\left[\frac{F_0 e^{i(\omega' t + \phi_0)}}{m\omega^2 - m\omega'^2 + \frac{im\omega\omega'}{Q}}\right] \\
&= \frac{F_0}{m\omega^2}\mathrm{Re}\left[\frac{e^{i(\omega' t + \phi_0)}}{1 - \frac{\omega'^2}{\omega^2} + \frac{i\omega'}{Q\omega}}\right] \\
&= \frac{F_0}{m\omega^2}\mathrm{Re}\left[\frac{1 - \frac{\omega'^2}{\omega^2} - \frac{i\omega'}{Q\omega}}{\left(1 - \frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2}\left(\cos\left(\omega' t + \phi_0\right) + i\sin\left(\omega' t + \phi_0\right)\right)\right] \\
&= \frac{F_0}{m\omega^2\left[\left(1 - \frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2\right]}\left(\cos\left(\omega' t + \phi_0\right)\left(1 - \frac{\omega'^2}{\omega^2}\right) + \sin\left(\omega' t + \phi_0\right)\frac{\omega'}{Q\omega}\right)
\end{aligned}$$

So we have

$$\begin{aligned}
x_0 = q(0) &= \frac{F_0}{m\omega^2}\mathrm{Re}\left[\frac{e^{i\phi_0}}{1 - \frac{\omega'^2}{\omega^2} + \frac{i\omega'}{Q\omega}}\right] \\
&= \frac{F_0\left[\cos\phi_0\left(1 - \frac{\omega'^2}{\omega^2}\right) + \sin\phi_0\frac{\omega'}{Q\omega}\right]}{m\omega^2\left[\left(1 - \frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2\right]} \\
v_0 = \dot{q}(0) &= \frac{F_0\omega'}{m\omega^2}\mathrm{Re}\left[\frac{ie^{i\phi_0}}{1 - \frac{\omega'^2}{\omega^2} + \frac{i\omega'}{Q\omega}}\right] \\
&= \frac{F_0\omega'\left[-\sin\phi_0\left(1 - \frac{\omega'^2}{\omega^2}\right) + \cos\phi_0\frac{\omega'}{Q\omega}\right]}{m\omega^2\left[\left(1 - \frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2\right]}
\end{aligned}$$

in order that there be no transient term.

(b) If $x_0 = 0$ and $v_0 = 0$, we then have

$$\begin{aligned}
q_h(0) &= \mathrm{Re}\left\{A \exp\left[i\left(\omega t + \phi\right)\right]\right\} \\
&= -q_p(0) \\
&= \mathrm{Re}\left[\frac{F_0 e^{i(\phi_0+\pi)}}{m\omega^2 - m\omega'^2 + \frac{im\omega\omega'}{Q}}\right] \\
&= \frac{F_0}{m\omega^2}\,\mathrm{Re}\left[\frac{e^{i(\phi_0+\pi)}}{1 - \frac{\omega'^2}{\omega^2} + \frac{i\omega'}{Q\omega}}\right] \\
&= \frac{F_0}{m\omega^2}\frac{1}{\sqrt{\left(1-\frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2}}\,\mathrm{Re}[e^{i(\phi_0+\pi+\psi)}]
\end{aligned}$$

where

$$\tan\psi = -\frac{1-\frac{\omega'^2}{\omega^2}}{\frac{\omega'}{Q\omega}}$$

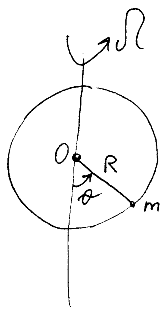
So the amplitude and phase are

$$\begin{aligned}
A &= \frac{F_0}{m\omega^2}\frac{1}{\sqrt{\left(1-\frac{\omega'^2}{\omega^2}\right)^2 + \left(\frac{\omega'}{Q\omega}\right)^2}} \\
\phi &= \phi_0 + \pi + \psi \\
&= \phi_0 + \pi - \arctan\left(\frac{1-\frac{\omega'^2}{\omega^2}}{\frac{\omega'}{Q\omega}}\right)
\end{aligned}$$

PROBLEM 3

PART A

(Bead on a hoop that rotates with angular velocity Ω about a vertical diameter)



Let's define spherical coordinates (r, θ, φ) with the center of the hoop being the origin.

We know from the previous homework #5 (Problem 4):

kinetic energy in spherical coord.: $\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \cdot \dot{\phi}^2\right)}$

(in fact, students need to memorize it)

(1) Constraints are given by:

$r = R$ (=const) ; $\phi = \Omega t$ because $\frac{d\phi}{dt} = \Omega$

(2) $\Rightarrow$ let's define functions $G_r = r - R = 0$,

$G_\phi = \phi - \Omega t = 0$.

Lagrange's function:

(3) $$\boxed{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \cdot \sin^2\theta \cdot \dot{\phi}^2\right) + mgr\cdot\cos\theta}$$

where potential energy is given by

$U = -mgr\cdot\cos\theta$.

(sign "minus" is due to the choice of angle θ in Hand & Finch: potential energy U must be the smallest for $\theta = 0$)

Let's now write the modified Euler-Lagrange equations (with Lagrange multipliers) which will allow us to calculate the constraint forces.

(4)
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} - \lambda_r \frac{\partial G_r}{\partial r} = 0;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{(no constraint in } \theta\text{)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} - \lambda_\varphi \frac{\partial G_\phi}{\partial \phi} = 0$$

Note: Factors of Ω corrected to Ω^2 in expressions for λ_r and N_r in handwritten page 4.

(3)

Explicitly:

$$(5)\quad \begin{cases} m\ddot r - mr\dot\theta^2 - mr\sin^2\theta\,\dot\varphi^2 - mg\cos\theta - \lambda_r = 0, \\ mr^2\sin\theta\cos\theta\,\dot\varphi^2 - mgr\sin\theta - 2mr\dot r\dot\theta - mr^2\ddot\theta = 0 \\ mr^2\sin^2\theta\,\ddot\varphi + 2mr^2\sin\theta\cos\theta\,\dot\theta\dot\varphi + 2mr\sin^2\theta\,\dot r\dot\varphi - \lambda_\varphi = 0 \end{cases}$$

Let's use constraints:

$r = R$; $\dot r = \ddot r = 0$; $\dot\varphi = \Omega$, $\ddot\varphi = 0$.

So, equations for r and φ are easy and simple.

$$(6)\quad \left[\begin{array}{l} r = R; \\ \varphi = \Omega t; \\ \ddot\theta - \Omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta = 0 \end{array}\right.$$

(the third eq. for θ is more difficult)

(The last eq. for θ follows from the equation on the top of this page,

$$r\ddot\theta + \cancel{2\dot r\dot\theta} + g\sin\theta - r\sin\theta\cos\theta\,\dot\varphi^2 = 0$$

Substitute $r = R$, $\dot r = 0$, $r = R$, $\dot\varphi = \Omega$

(4)

Lagrange multipliers λ_r and λ_φ allow us to calculate the constraint forces.

From eq. (5) for r on the previous page:

$$\lambda_r = -mR\dot\theta^2 - mR\sin^2\theta\,\Omega^2 - mg\cos\theta$$

$$N_r = \sum_\ell \lambda_\ell \cdot \frac{\partial G_r}{\partial r} = \lambda_r \cdot \frac{\partial G_r}{\partial r} = \lambda_r \cdot 1 = \lambda_r.$$

$$\boxed{N_r = -mR\dot\theta^2 - mR\Omega^2\sin^2\theta - mg\cos\theta}$$

The first two terms in this expression are centripetal terms (forces) for the circular motions in θ and φ respectively (in other words, the first term $\sim\dot\theta^2$ is due to the hoop's rotation about its diameter and the second term is due to the motion of the bead around the hoop).

The third term $-mg\cos\theta$ is due to gravity. It is $|\vec F| = mg$ projected onto $\hat r$.

(5)

From eq.(5) for ϕ:

$$\lambda_\phi = 2mR^2\Omega\cdot\sin\theta\cdot\cos\theta\cdot\dot\theta,$$

and $N_\phi = \lambda_\phi\cdot\frac{\partial G_\phi}{\partial\phi} = \lambda_\phi$,

$$\boxed{N_\phi = 2mR^2\Omega\cdot\sin\theta\cdot\cos\theta\cdot\dot\theta}$$

N_ϕ depends on $\dot\theta$ because changing angle θ changes the radius of the circle about the vertical diameter the bead would make if θ would be constant.

In other words, $N_\phi \sim \frac{d}{dt}P_\phi$, and P_ϕ depends on θ.

(6)

Problem 3, part (b)

Let's start from equation for θ

$$(7)\quad \boxed{\ddot\theta - \Omega^2\cdot\sin\theta\cos\theta + \frac{g}{R}\cdot\sin\theta = 0}$$

Remark

We could, in fact, quickly obtain it from Lagrangian L with constant values of $r=R$ and $\dot\varphi=\Omega$:

$$L=\frac{m}{2}\left(R^2\dot\theta^2 + R^2\sin^2\theta\,\Omega^2\right) + mgR\cos\theta;$$

$$\Rightarrow mR^2\ddot\theta - mR^2\sin\theta\cos\theta\,\Omega^2 + mgR\sin\theta = 0,$$

and we arrive at (7)

(i) Find equilibrium values of θ:

in equilibrium $\ddot\theta = 0 \Rightarrow$

$$\sin\theta_{eq}\cdot\left(R\Omega^2\cdot\cos\theta_{eq} - g\right) = 0;$$

$$R\cdot\sin\theta_{eq}\cdot\left(\Omega^2\cdot\cos\theta_{eq} - \frac{g}{R}\right) = 0$$

Consider two cases: 1°. $\Omega < \sqrt{\frac{g}{R}}$ or $\Omega^2\cdot\cos\theta_{eq} < \frac{g}{R}$

then $\left(\Omega^2\cdot\cos\theta_{eq} - \frac{g}{R}\right) < 0$ ← always nonzero $\Rightarrow \Rightarrow$ two θ_{eq}: $\boxed{\theta=0}$ or $\boxed{\theta=\pi}$

(7)

So, if $\Omega < \sqrt{\frac{g}{R}}$ there are two equilibrium positions $\theta_{eq} = 0$ or $\theta_{eq} = \pi$.

2°) if $\Omega > \sqrt{\frac{g}{R}}$, then in addition to $\theta_{eq} = 0$ and $\theta_{eq} = \pi$ we have now $\boxed{\cos\theta_{eq} = \frac{g}{R\Omega^2}}$

So, there are now three possible equilibrium positions

$$\boxed{\begin{array}{l}\theta = 0,\ \theta = \pi, \\ \theta = \arccos\frac{g}{R\Omega^2}\end{array}}$$

$\leftarrow$ existence of this value depends on the angular speed Ω.

(ii) Consider the stability of these θ_{eq}

For this purpose define $\theta = \theta_{eq} + \eta$, where η is a small quantity.

Keeping the first-order terms:

$$R\ddot{\eta} + (g\cdot\cos\theta_{eq} - R\Omega^2\cos 2\theta_{eq})\eta - R\Omega^2 \sin\theta_{eq}\cos\theta_{eq} + g\cdot\sin\theta_{eq} = 0$$

(8)

in the equilibrium position. $\ddot{\theta} = 0$

$\Rightarrow$ according to EOM (7) $R\Omega^2\sin\theta\cos\theta + g\cdot\sin\theta = 0$

$\Rightarrow$ we have $\ddot{\eta} + \left(\frac{g}{R}\cos\theta_{eq} - \Omega^2\cos 2\theta_{eq}\right)\eta = 0$.

If $\Omega < \sqrt{\frac{g}{R}} \Rightarrow$ (1) Coeff. in the brackets (parantheses) is positive for $\theta_{eq} = 0$.

$\boxed{\text{Stable}}$

(2) Coefficient in the brackets in negative for $\theta_{eq} = \pi$.

$\boxed{\text{Unstable}}$

If $\Omega > \sqrt{\frac{g}{R}} \Rightarrow$ Both $\boxed{\theta_{eq} = 0}$ and $\boxed{\theta_{eq} = \pi}$ become $\boxed{\text{Unstable}}$

In other words, the bottom position can be stable, but the upper position ($\theta = \pi$) cannot be stable

If $\Omega > \sqrt{\frac{g}{R}} \Rightarrow$ there is the third equilibrium position $\cos\theta_{eq} = \frac{g}{R\Omega^2}$

(9)

$$\frac{g}{R}\cdot\cos\theta_{eq} - 2\Omega^2\cos^2\theta_{eq} + \Omega^2 = \frac{g}{R}\cdot\frac{g}{R\Omega^2} - 2\Omega^2\cdot\frac{g^2}{R^2\Omega^4} + \Omega^2,$$

OR $$\frac{g^2}{R^2\Omega^2} - 2\frac{g^2}{R^2\Omega^2} + \Omega^2 = \frac{1}{\Omega^2}\left(\Omega^4 - \frac{g^2}{R^2}\right) > 0$$

$\boxed{\text{Stable}}$ (exists only if $\Omega > \sqrt{\frac{g}{R}}$)

→ Calculation of frequencies of small oscillations

$$\cos\theta = \cos(\theta_{eq} + \eta) = \cos\theta_{eq} - \eta\sin\theta_{eq} + O(\eta^2)$$

$$\sin\theta = \sin(\theta_{eq} + \eta) = \sin\theta_{eq} + \eta\cos\theta_{eq} + O(\eta^2)$$

$$\sin\theta\cdot\cos\theta = \cos\theta_{eq}\sin\theta_{eq} - \eta\sin^2\theta_{eq} + \eta\cos^2\theta_{eq} + \cancel{\eta}^2$$

therefore $$\sin\theta\cos\theta = \cos\theta_{eq}\cdot\sin\theta_{eq} + \eta\cdot\cos 2\theta_{eq},$$

$$0 = \ddot{\eta} + \frac{g}{R}\cos\theta_{eq}\cdot\eta - \Omega^2\cos 2\theta_{eq}\,\eta + \underbrace{\left(\frac{g}{R}\sin\theta_{eq} - \Omega^2\cos\theta_{eq}\sin\theta_{eq}\right)}_{\equiv 0}$$

⇒ FREQUENCY OF SMALL OSCILLATIONS IS: $$\boxed{\omega = \sqrt{\frac{g}{R}\cdot\cos\theta_{eq} - \Omega^2\cdot\cos 2\theta_{eq}}}$$

(10)

This expression gives the following results of frequencies for small oscillations:

$$\omega = \sqrt{\frac{g}{R}\cos\theta_{eq} - \Omega^2\cos 2\theta_{eq}} =$$

$$= \begin{cases} \sqrt{\frac{g}{R} - \Omega^2} & \text{FOR } \theta_{eq} = 0 \\ \left(\sqrt{\Omega^4 - \left(\frac{g}{R}\right)^2}\Big/\Omega\right) & \text{FOR } \theta_{eq} = \operatorname{Arccos}\frac{g}{R\Omega^2} \end{cases}$$

Note: New solution to 3(b)(iii), 2006/12/03.

Problem 3, part (b)(iii)

Frictional force $F_{friction} = -bv$;

$$V = R\cdot\dot{\theta} \Rightarrow F_{friction} = -bR\dot{\theta}.$$

We need to substitute this force into the equation for θ (or, equivalently into eq. for η)

$$mR\ddot{\theta} + F_{friction} - mR\sin\theta\cdot\cos\theta\cdot\Omega^2 + mgR\cdot\sin\theta = 0$$

or $$\ddot{\theta} + \frac{F_{friction}}{mR} + [\text{some function of } \theta] = 0.$$

$$\ddot{\theta} - \frac{b}{m}\dot{\theta} + [\text{some function of } \theta] = 0$$

$$Q \equiv \omega\cdot\tau_{damp}, \quad \tau_{damp} = m/b.$$

(compare with the upper equation on page 170 of lecture notes)

So, $\boxed{Q = \frac{\omega\cdot m}{b}}$

And we can substitute expressions for frequencies ω from part (b) of this ~~problem~~ problem.

(1)

Problem 4

Green's function for the underdamped oscillator is given in the lecture notes (page 177, eq 3.10)

$$G(t-t') = \frac{1}{\omega'} \cdot \exp\left[-\frac{t-t'}{2Q}\right] \sin[\omega'(t-t')] \cdot \theta(t-t')$$

The driving force is $F(t) = F_0(1-e^{-\alpha t})\,\underbrace{\theta(t)}_{\text{means } F(t)\equiv 0 \text{ for } t<0}$, the response is

$$q(t) = \int_{-\infty}^{\infty} dt' \cdot F(t')\, G(t-t') =$$

$$= \frac{F_0}{\omega'} \int_0^t e^{-\frac{(t-t')}{2Q}} \cdot \sin[\omega'(t-t')]\left(1-e^{-\alpha t'}\right) dt' =$$

$$= \underbrace{\frac{F_0}{\omega'} \int_0^t e^{-\frac{(t-t')}{2Q}} \cdot \sin[\omega'(t-t')]\, dt'}_{I_1} - \frac{F_0}{\omega'} \int_0^t e^{-\frac{(2Q\alpha-1)t'}{2Q} - \frac{t}{2Q}} \times \sin[\omega'(t-t')]\, dt' = I_2$$

(2)

There are two ways of calculating the integral of the type $\int_0^a e^{bx} \sin \omega x \, dx$

One method is: complex variables

$$\sin \omega x = \mathrm{Re}[-ie^{i\omega x}]$$

And the second method: apply two times integration by parts.

Let's consider method of complex variables.

$$I_1 = \int_0^t e^{-\frac{t-t'}{2Q}} \cdot \sin[\omega'(t-t')]\, dt' = \mathrm{Re} \int_0^t \left(-i e^{-\frac{t-t'}{2Q} + i\omega'(t-t')}\right) dt' =$$

$$= \mathrm{Re}\left[-i \frac{1}{\frac{1}{2Q} - i\omega'} e^{-\frac{t-t'}{2Q} + i\omega'(t-t')}\right]\Bigg|_{0}^{t} =$$

(t' limits)

$$= \mathrm{Re}\left[\frac{-i\left(\frac{1}{2Q} + i\omega'\right)}{\left(\frac{1}{2Q}\right)^2 + (\omega')^2} \times \left(1 - e^{-\frac{t}{2Q} + i\omega' t}\right)\right] =$$

$$= \frac{\omega'}{\left(\frac{1}{2Q}\right)^2 + (\omega')^2} \times \left(1 - e^{-\frac{t}{2Q}} \cdot \cos \omega' t\right) —$$

$$-\frac{1}{2Q\left[\left(\frac{1}{2Q}\right)^2+(\omega')^2\right]}\times \exp\left[-\frac{t}{2Q}\right]\times \sin\omega' t\,;$$

3

Now consider the second integral:

$$I_2=\int_0^t e^{-\frac{(2Q\alpha-1)t'}{2Q}-\frac{t}{2Q}}\times \sin\left[\omega'(t-t')\right]dt'=$$

$$=\mathrm{Re}\left[\int_0^t -i\cdot \exp\left[-\alpha t'-\frac{t-t'}{2Q}+i\omega'(t-t')\right]dt'\right]=$$

$$=\mathrm{Re}\left[\frac{-i}{-\alpha+\frac{1}{2Q}-i\omega'}\cdot \exp\left[-\alpha t'-\frac{t-t'}{2Q}+i\omega'(t-t')\right]\Bigg|_0^t\right]$$

$$\frac{-i\left(-\alpha+\frac{1}{2Q}+i\omega'\right)}{\left(-\alpha+\frac{1}{2Q}\right)^2+(\omega')^2}\times\left(e^{-\alpha t}-e^{-\frac{t}{2Q}+i\omega' t}\right)=$$

$$=\frac{\omega'}{\left(\frac{1}{2Q}-\alpha\right)^2+(\omega')^2}\times\left[e^{-\alpha t}-e^{-\frac{t}{2Q}}\cdot\cos\omega' t\right]+$$

$$+\frac{\alpha-\frac{1}{2Q}}{\left(\frac{1}{2Q}-\alpha\right)^2+(\omega')^2}\times e^{-\frac{t}{2Q}}\cdot\sin\omega' t\,;$$

Therefore,

$$q(t)=\frac{F_0}{\omega'}I_1-\frac{F_0}{\omega'}I_2=$$

$$=F_0\cdot\frac{1}{\left(\frac{1}{2Q}\right)^2+(\omega')^2}\cdot\left(1-e^{-\frac{t}{2Q}}\cdot\cos\omega' t\right)-$$

$$-\left(\frac{F_0}{2Q\omega'}\right)\times\left(\frac{1}{\left(\frac{1}{2Q}\right)^2+(\omega')^2}\right)\times e^{-\frac{t}{2Q}}\times\sin\omega' t\;-$$

$$-\frac{F_0}{\left(\frac{1}{2Q}-\alpha\right)^2+(\omega')^2}\cdot\left(e^{-\alpha t}-e^{-\frac{t}{2Q}}\cdot\cos\omega' t\right)-$$

$$-F_0\cdot\frac{\alpha-\frac{1}{2Q}}{\left(\frac{1}{2Q}-\alpha\right)^2+(\omega')^2}\cdot e^{-\frac{t}{2Q}}\cdot\sin\omega' t$$

Problem 5

For a conservative system, the Hamiltonian $H(q_i, p_i)$ is independent of t. Consider a canonical transformation in which the new coordinates are $\gamma_i(\alpha_k)$, independent functions of α_k, where α_k are constants of the motion and, in particular, α_1 is the constant of motion, H. If the generating function for this transformation be denoted by $W(p, Q)$, then the equations of the transformation are

$$q_i = -\frac{\partial W}{\partial p_i}, \;\; P_i = -\frac{\partial W}{\partial Q_i} = -\frac{\partial W}{\partial \gamma_i}$$

The condition now determining $W(p, \gamma_i(\alpha_k))$ is that H shall be equal to α_1 for the conservative system

$$\begin{aligned} H(q_i, p_i) + \frac{\partial W}{\partial t} &= H(q_i, p_i) = \alpha_1 \\ &\Rightarrow H\left(-\frac{\partial W}{\partial p_i}, p_i\right) = \alpha_1 \end{aligned}$$

where we use the fact W is independent of t and α_1 is a function of new coordinates are γ_i. The equations of motion are

$$\begin{aligned} \dot{P}_i &= -\frac{\partial H}{\partial Q_i} = -\frac{\partial H}{\partial \gamma_i} = -v_i(\gamma_k) \Rightarrow P_i = v_i t + \beta_i \\ \dot{Q}_i &= \frac{\partial H}{\partial P_i} = 0 \qquad Q_i = \gamma_i \end{aligned}$$

The equations

$$P_i = v_i(\gamma_k)\, t + \beta_i = -\frac{\partial W(p, \gamma_i(\alpha_k))}{\partial \gamma_i}$$

can be "turned inside out" to furnish p_i in terms of α, β and t :

$$p_i = p_i(\alpha, \beta, t) \tag{3}$$

After the differentiation in

$$q_i = -\frac{\partial W(p, \gamma_i(\alpha_k))}{\partial p_i}$$

has been performed, Eq. (3) may be substituted for the $p's$, thus giving q_i as functions of α, β and t :

$$q_i = q_i(\alpha, \beta, t)$$

We can use

$$q_i|_{t=0} = -\frac{\partial W(p, \gamma_i(\alpha_k))}{\partial p_i}|_{t=0}$$

to express α_i or $\gamma_i(\alpha_k)$ in terms of the initial conditions $q_i|_{t=0}$ and $p_i|_{t=0}$. The constants β_i can be obtained from the initial conditions by

$$P_i = v_i(\gamma_k)\, t + \beta_i = -\frac{\partial W(p, \gamma_i(\alpha_k))}{\partial \gamma_i}|_{t=0}$$