Physics 106a/196a – Problem Set 7 – Due Dec 1, 2006
Solutions

Andrey Rodionov, Peng Wang, Sunil Golwala

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Version 2: Less kludgey, more exhaustive solution for Problem 5. **Version 3:** Minor addition to PS1 solution.

Problem 1

1

PROBLEM 1. (Box sliding horizontally)

Let us use the Lagrangian from Homework 3:

$$L = T - U = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\dot{x}^2 + 2 l \cos\theta \, \dot{x} \dot{\theta} + l^2 \dot{\theta}^2 \right] + mgl\cos\theta$$

(M is the mass of the box, l is the pendulum's length)

We are considering small oscillations therefore we need to Taylor expand trigonometric functions in $L = T - U$ around the position of stable equilibrium $\theta = 0$.

$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\dot{x}^2 + 2 l \dot{x} \dot{\theta} + l^2 \dot{\theta}^2 \right];$$

$$U = -mgl\left(1 - \frac{\theta^2}{2}\right);$$

(2)

Matrix $\hat{t}_{ij} = \frac{1}{2}\frac{\partial^2 T}{\partial \dot{q}_i \cdot \partial \dot{q}_j} = \frac{1}{2}\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}$;

Matrix $\hat{V}_{ij} = \frac{1}{2}\frac{\partial^2 U}{\partial q_i \cdot \partial q_j} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$

Equation for normal mode frequencies:

$$\det\left|\hat{V} - \omega^2 \cdot \hat{t}\right| = 0;$$

$$\frac{1}{2}\cdot\left\|\begin{matrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & mgl - \omega^2 ml^2 \end{matrix}\right\| = 0;$$

(3)

$$\omega^4 Mml^2 + \omega^4 m^2 l^2 - \omega^2 Mmgl - \omega^2 m^2 gl - \omega^4 m^2 l^2 = 0;$$

$$Mml^2 \cdot \omega^4 - (Mg \cdot ml + m^2 g l)\,\omega^2 = 0;$$

$$\boxed{Ml\omega^4 - (Mg + mg)\,\omega^2 = 0.}$$

Solutions for ω: $\boxed{\omega_1^2 = 0}$

or $\boxed{\omega_2^2 = \frac{(M+m)\,g}{Ml}}$

ω_1 corresponds to the trivial case when the box isn't oscillating, purely translational motion of box.

→ Now: find normal mode vectors.

A) $\left(-\omega_1^2 \cdot \hat{t} + \hat{V}\right)\underbrace{\begin{pmatrix} x \\ \theta \end{pmatrix}}_{\text{call this } \Phi_1} = 0$, or $\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = 0$

Therefore, $\boxed{\Phi_1 = \alpha_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$. Angle θ is always ~~zero~~. horizontal coord. x can take any value.

(4)

b) Normal mode vectors for $\omega_2 \neq 0$:

$$\left(-\omega_2^2 \hat{t} + \hat{V}\right)\begin{pmatrix} X \\ \theta \end{pmatrix} = 0 \;; \quad \text{call } \Phi_2 \equiv \begin{pmatrix} X \\ \theta \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} -\frac{(M+m)(M+m)\cdot g}{Ml} & -\frac{(M+m)\cdot ml\cdot g}{M\cdot l} \\ -\frac{(M+m)\cdot ml\cdot g}{M\cdot l} & -\frac{(M+m)\cdot ml^2\cdot g}{Ml} + mgl \end{pmatrix}\begin{pmatrix} X \\ \theta \end{pmatrix}$$

Or:

$$\begin{pmatrix} \frac{(M+m)^2}{Ml} & \frac{(M+m)m}{M} \\ \frac{(M+m)m}{M} & \frac{(M+m)ml}{M} - \frac{mgl}{g} \end{pmatrix}\begin{pmatrix} X \\ \theta \end{pmatrix} = 0$$

This reduces to $X = -\frac{ml}{M+m}\theta$.

Therefore, $\boxed{\Phi_2 = \alpha_2 \begin{pmatrix} -\frac{ml}{M+m} \\ 1 \end{pmatrix}}$

(5)

→ Normalization Condition.

$1 = \Phi^T \hat{t} \Phi$;

as $\Phi_2 = \alpha_2 \begin{pmatrix} -\frac{ml}{M+m} \\ 1 \end{pmatrix}$, $\Phi_2^T = \alpha_2^* \begin{pmatrix} -\frac{ml}{M+m} & 1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \frac{\alpha_2^2}{2}\begin{pmatrix} -\frac{ml}{M+m} & 1 \end{pmatrix}\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} -\frac{ml}{M+m} \\ 1 \end{pmatrix} = 1;$$

This leads to a condition

$$\boxed{\alpha_2 = \sqrt{\frac{2\cdot(M+m)}{-m^2l^2 + ml^2(M+m)}}}.$$

Let us normalize also $\vec{\Phi}_1$: $\vec{\Phi}_1 = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Phi_1^T \hat{t} \Phi_1 = \frac{1}{2}(\alpha_1 \;\; 0)\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \frac{1}{2}(\alpha_1 \;\; 0)\begin{pmatrix} (M+m)\alpha_1 \\ ml\alpha_1 \end{pmatrix}$$

$= \frac{1}{2}(M+m)\alpha_1^2 = 1$; $\boxed{\alpha_1 = \sqrt{\frac{2}{M+m}}}$

$$\alpha_2 = \sqrt{\frac{2(M+m)}{-m^2l^2 + (M+m)ml^2}} = \sqrt{\frac{2(M+m)}{Mml^2}}$$

Problem 2

At rest, the spring has a length l_s. The coordinates of the two ends of the spring are

$$\left(\frac{l_s}{2}+\frac{l}{2}\sin\theta_2,-\frac{l}{2}\cos\theta_2\right)$$
$$\left(-\frac{l_s}{2}+\frac{l}{2}\sin\theta_1,-\frac{l}{2}\cos\theta_1\right)$$

with the origin chosen at the midpoint between the pivots of the two pendulums. And the velocities of the two ends are

$$\left(\frac{l}{2}\cos\theta_2\dot{\theta}_2,\frac{l}{2}\sin\theta_2\dot{\theta}_2\right)$$
$$\left(\frac{l}{2}\cos\theta_1\dot{\theta}_1,\frac{l}{2}\sin\theta_1\dot{\theta}_1\right)$$

The length of the spring in terms of θ_1 and θ_2 is

$$\begin{aligned} l_s\left(\theta_1,\theta_2\right) &= \sqrt{\left[l_s+\frac{l}{2}\left(\sin\theta_2-\sin\theta_1\right)\right]^2+\frac{l^2}{4}\left(\cos\theta_1-\cos\theta_2\right)^2} \\ &\sim l_s+\frac{l}{2}\left(\sin\theta_2-\sin\theta_1\right)+O\left(\theta^4\right) \end{aligned}$$

where we keep terms up to $O\left(\theta^2\right)$ in the second line. Let α be a parameter that describes the position along the spring, $0\leqslant\alpha\leqslant l_s\left(\theta_1,\theta_2\right)$. The coordinates of the spring as a function of the angles and the parameter α are

$$\begin{aligned} &\begin{pmatrix} \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(\frac{l_s}{2}+\frac{l}{2}\sin\theta_2\right)+\left(1-\frac{\alpha}{l_s(\theta_1,\theta_2)}\right)\left(-\frac{l_s}{2}+\frac{l}{2}\sin\theta_1\right), \\ \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(-\frac{l}{2}\cos\theta_2\right)+\left(1-\frac{\alpha}{l_s(\theta_1,\theta_2)}\right)\left(-\frac{l}{2}\cos\theta_1\right) \end{pmatrix} \\ =&\begin{pmatrix} \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(l_s+\frac{l}{2}\left(\sin\theta_2-\sin\theta_1\right)\right)+\left(-\frac{l_s}{2}+\frac{l}{2}\sin\theta_1\right), \\ \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(\frac{l}{2}\left(\cos\theta_1-\cos\theta_2\right)\right)+\left(-\frac{l}{2}\cos\theta_1\right) \end{pmatrix} \end{aligned}$$

and the velocity of the spring is

$$\begin{aligned} &\begin{pmatrix} \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(\frac{l}{2}\cos\theta_2\dot{\theta}_2\right)+\left(1-\frac{\alpha}{l_s(\theta_1,\theta_2)}\right)\left(\frac{l}{2}\cos\theta_1\dot{\theta}_1\right), \\ \frac{\alpha}{l_s(\theta_1,\theta_2)}\left(\frac{l}{2}\sin\theta_2\dot{\theta}_2\right)+\left(1-\frac{\alpha}{l_s(\theta_1,\theta_2)}\right)\left(\frac{l}{2}\sin\theta_1\dot{\theta}_1\right) \end{pmatrix} \\ \sim&\left(\frac{\alpha}{l_s}\left(\frac{l}{2}\dot{\theta}_2\right)+\left(1-\frac{\alpha}{l_s}\right)\left(\frac{l}{2}\dot{\theta}_1\right),0\right)+O\left(\theta^2\right) \end{aligned}$$

where in the last line we neglect the higher order terms than $O\left(\theta\right)$ for we use this expansion to calculate the kinetic energy where we have to square the time derivative of the coordinates. The gravity potential energy of the spring is

$$\begin{aligned}
U_s &= \int_0^{l_s(\theta_1,\theta_2)} d\alpha \frac{m_s g}{l_s(\theta_1,\theta_2)} \left[\frac{\alpha}{l_s(\theta_1,\theta_2)}\left(\frac{l}{2}(\cos\theta_1 - \cos\theta_2)\right) + \left(-\frac{l}{2}\cos\theta_1\right)\right] \\
&= \frac{m_s g}{l_s(\theta_1,\theta_2)} \left[\frac{l_s^2(\theta_1,\theta_2)}{2l_s(\theta_1,\theta_2)}\left(\frac{l}{2}(\cos\theta_1 - \cos\theta_2)\right) + l_s(\theta_1,\theta_2)\left(-\frac{l}{2}\cos\theta_1\right)\right] \\
&= m_s g\left[\frac{1}{2}\left(\frac{l}{2}(\cos\theta_1 - \cos\theta_2)\right) + \left(-\frac{l}{2}\cos\theta_1\right)\right] \\
&\sim \frac{m_s g l}{2}\left[\frac{1}{4}\left(-\theta_1^2 + \theta_2^2\right) + \frac{\theta_1^2}{2}\right] = \frac{m_s g l}{8}\left(\theta_1^2 + \theta_2^2\right)
\end{aligned}$$

The kinetic energy of the spring is

$$\begin{aligned}
T_s &= \int_0^{l_s(\theta_1,\theta_2)} d\alpha \frac{1}{2}\frac{m_s}{l_s(\theta_1,\theta_2)}\left(\dot{x}^2 + \dot{y}^2\right) \\
&= \frac{m_s}{2l_s(\theta_1,\theta_2)} \int_0^{l_s(\theta_1,\theta_2)} d\alpha \left(\frac{\alpha}{l_s}\left(\frac{l}{2}\dot{\theta}_2\right) + \left(1 - \frac{\alpha}{l_s}\right)\left(\frac{l}{2}\dot{\theta}_1\right)\right)^2 \\
&= \frac{m_s l^2}{24}\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)
\end{aligned}$$

The Lagrangian has to include the spring's part

$$\begin{aligned}
L_s &= T_s - U_s \\
&= \frac{m_s l^2}{24}\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) - \frac{m_s g l}{8}\left(\theta_1^2 + \theta_2^2\right)
\end{aligned}$$

The kinetic energy and potential energy matrices are

$$\mathbf{t} = \begin{pmatrix} \frac{ml^2}{2} + \frac{m_s l^2}{24} & \frac{m_s l^2}{48} \\ \frac{m_s l^2}{48} & \frac{ml^2}{2} + \frac{m_s l^2}{24} \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} \frac{mgl}{2} + \frac{kl^2}{8} + \frac{m_s gl}{8} & -\frac{kl^2}{8} \\ -\frac{kl^2}{8} & \frac{mgl}{2} + \frac{kl^2}{8} + \frac{m_s gl}{8} \end{pmatrix}$$

Finding the normal mode frequencies is simply a matter of solving the determinant equation $\left|-\omega^2\mathbf{t} + \mathbf{v}\right| = 0$. We thus have

$$\frac{mgl}{2}\begin{vmatrix} 1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega^2\left(\frac{l}{g} + \frac{m_s l}{6mg}\right) & -\frac{kl}{4mg} - \omega^2\frac{m_s l}{24mg} \\ -\frac{kl}{4mg} - \omega^2\frac{m_s l}{24mg} & 1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega^2\left(\frac{l}{g} + \frac{m_s l}{6mg}\right) \end{vmatrix} = 0$$

$$1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega^2\left(\frac{l}{g} + \frac{m_s l}{6mg}\right) = \pm\left(\frac{kl}{4mg} + \omega^2\frac{m_s l}{24mg}\right)$$

$$\omega^2 = \frac{1 + \frac{kl}{4mg} + \frac{m_s}{4m} \mp \frac{kl}{4mg}}{\frac{l}{g} + \frac{m_s l}{6mg} \pm \frac{m_s l}{24mg}}$$

The normal mode frequencies are therefore

$$\omega_1^2 = \frac{1 + \frac{m_s}{4m}}{\frac{l}{g} + \frac{5m_s l}{24mg}}$$

$$\omega_2^2 = \frac{1 + \frac{kl}{2mg} + \frac{m_s}{4m}}{\frac{l}{g} + \frac{m_s l}{8mg}}$$

To find the mode amplitude ratios, we can do it directly by solving the equation

$$\left(-\omega_i^2 \mathbf{t} + \mathbf{v}\right) \overrightarrow{\Phi}_i = 0$$

$$\begin{pmatrix} 1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega^2 \left(\frac{l}{g} + \frac{m_s l}{6mg}\right) & -\frac{kl}{4mg} - \omega^2 \frac{m_s l}{24mg} \\ -\frac{kl}{4mg} - \omega^2 \frac{m_s l}{24mg} & 1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega^2 \left(\frac{l}{g} + \frac{m_s l}{6mg}\right) \end{pmatrix} \overrightarrow{\Phi}_i = 0$$

$$\frac{\Phi_{i,1}}{\Phi_{i,2}} = \frac{\frac{kl}{4mg} + \omega_i^2 \frac{m_s l}{24mg}}{1 + \frac{kl}{4mg} + \frac{m_s}{4m} - \omega_i^2 \left(\frac{l}{g} + \frac{m_s l}{6mg}\right)}$$

which give for the two cases

$$\frac{\Phi_{1,1}}{\Phi_{1,2}} = 1$$

$$\frac{\Phi_{2,1}}{\Phi_{2,2}} = -1$$

Problem 3

Problem 3 ((Satellite is launched at an angle α with the vertical, with a velocity v_0. (1)

A) Firstly, write all energies:

Initial energy:

$$E_0 = \frac{1}{2} m v_0^2 - gmR \left(= \frac{1}{2} m v_0^2 - \frac{G M_{earth} m}{R} \right)$$

$$g = \frac{G M_{EARTH}}{R^2}$$

$$E_1 = \frac{1}{2} m v_1^2 - mgR \cdot \frac{R}{R+h_1} \rightarrow \text{energy before the thrust,}$$

$$\ell_1 = m v_1 \cdot (R+h_1) \rightarrow \text{angular momentum before the thrust;}$$

Energy after the thrust is applied, so that the additional velocity Δv_1 is added, is given by

$$E_{fin} = -\frac{GmM}{2R+h_1+h_2} = -mg \frac{R^2}{2R+h_1+h_2};$$

Angular momentum after the thrust: $\ell_{fin} = m(v_1 + \Delta v_1)(R+h_1)$;

B) Apply the energy conservation to find v_0

From the conservation of energy:

$$E_0 = E_1 \Rightarrow \frac{1}{2} m v_0^2 - mgR = \frac{1}{2} m v_1^2 - mg\left(\frac{R}{R+h_1}\right) \cdot R;$$

which is equivalent to (2)

$$(1) \quad V_1^2 = V_0^2 - 2gR + \frac{2gR^2}{R+h_1} = v_0^2 + \underbrace{2gR\left(\frac{R}{R+h_1} - 1\right)}_{<0 \text{ negative}}$$

From the other point of view, V_1 can be expressed in terms of V_0 and $\sin\alpha$ using the law of conservation of angular momentum (before the thrust, consider only the initial moment and the moment before the thrust):

$$\ell_0 = \ell_1, \text{ or } \quad m V_0 R \cdot \sin\alpha = m v_1 (R + h_1)$$

$$(2) \quad \Rightarrow \quad v_1 = \frac{R \cdot \sin\alpha}{R+h_1} v_0;$$

Substitute (2) into (1) $\Rightarrow$

$$v_0^2 \left(\frac{R \cdot \sin\alpha}{R+h_1}\right)^2 = V_0^2 + 2gR\left(\frac{R}{R+h_1} - 1\right);$$

(3)

$$V_0^2 = \frac{2gR\left(1-\frac{R}{R+h_1}\right)}{1-\left(\frac{R\cdot\sin\alpha}{R+h_1}\right)^2};$$

$$\boxed{V_0 = \sqrt{\frac{2gRh_1}{(R+h_1)\left(1-\frac{R^2\sin^2\alpha}{(R+h_1)^2}\right)}}}$$

(c) Apply the <u>energy conservation</u> to find Δv_1:
In order to find the necessary Δv_1 apply: $E_{fin} = E_1$, "with v_1 replaced by $v_1+\Delta v_1$", i.e.

$$-mgR\left(\frac{R}{2R+h_1+h_2}\right) = -mgR\cdot\frac{R}{R+h_1} + \frac{1}{2}m(v_1+\Delta v_1)^2;$$

so

$$(v_1+\Delta v_1)^2 = -2gR\cdot\left[\frac{R}{2R+h_1+h_2} - \frac{R}{R+h_1}\right];$$

$$\Delta v_1 = \sqrt{2gR\underbrace{\left(\frac{R}{R+h_1} - \frac{R}{2R+h_1+h_2}\right)}} - v_1$$

$R+h_1 < 2R+h_1+h_2 \Rightarrow$ this expression is <u>not</u> negative

(d)

We can make use of formula (1) $\Rightarrow$
$\Rightarrow$ substitute expression for V_1 in terms of V_0, and then use our result for V_0.

$$\Delta v_1 = \sqrt{2gR}\left[\sqrt{\frac{R}{R+h_1} - \frac{R+h_1}{2R+h_1+h_2}} - \sqrt{\frac{V_0^2}{2gR} + \left(\frac{R}{R+h_1}-1\right)}\right], \text{ and}$$

then

$$\Delta v_1 = \sqrt{2gR}\left[\sqrt{\frac{R}{R+h_1} - \frac{R+h_1}{2R+h_1+h_2}} - \sqrt{\frac{\cancel{R}\,h_1}{(R+h_1)\cdot\left(1-\frac{R^2\sin^2\alpha}{(R+h_1)^2}\right)} + \left(\frac{R}{R+h_1}-1\right)}\right].$$

Problem 4

The kinetic energy and potential energy matrices are

$$\mathbf{t}=\frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$\mathbf{v}=\frac{1}{2}\begin{pmatrix}1 & -\epsilon & -\epsilon\\ -\epsilon & 1 & -\epsilon\\ -\epsilon & -\epsilon & 1\end{pmatrix}$$

Finding the normal mode frequencies is to solve the determinant equation $\left|-\omega^2\mathbf{t}+\mathbf{v}\right|=0$. We thus have

$$\begin{vmatrix}1-\omega^2 & -\epsilon & -\epsilon\\ -\epsilon & 1-\omega^2 & -\epsilon\\ -\epsilon & -\epsilon & 1-\omega^2\end{vmatrix}=0$$
$$\left(1-\omega^2\right)\left[\left(1-\omega^2\right)^2-\epsilon^2\right]+\epsilon\left[-\epsilon\left(1-\omega^2\right)-\epsilon^2\right]-\epsilon\left[\epsilon^2+\epsilon\left(1-\omega^2\right)\right]=0$$
$$\left(1-\omega^2\right)^3-3\left(1-\omega^2\right)\epsilon^2-2\epsilon^3=0$$

The normal mode frequencies are therefore

$$\omega_1^2=1-2\epsilon$$
$$\omega_{2,3}^2=1+\epsilon$$

Since $\omega^2>0$, one has

$$-1<\epsilon<\frac{1}{2}$$

If we try to find the normal mode vectors via the usual cofactor vectors, we find

$$\overrightarrow{\Phi}_i=\alpha\begin{pmatrix}\left(1-\omega_i^2\right)^2-\epsilon^2\\ \epsilon\left(1-\omega_i^2\right)+\epsilon^2\\ \epsilon^2+\epsilon\left(1-\omega_i^2\right)\end{pmatrix}$$

For $\omega_1^2=1-2\epsilon$, the mode vector is

$$\overrightarrow{\Phi}_1=3\epsilon^2\alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$$

where the normalization condition gives us

$$\overrightarrow{\Phi}_1^T\mathbf{t}\overrightarrow{\Phi}_1=1\Rightarrow\left(3\epsilon^2\alpha\right)^2=\frac{2}{3}$$

So the $\omega_1^2=1-2\epsilon$ mode vector is

$$\overrightarrow{\Phi}_1=\sqrt{\frac{2}{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

This mode is the oscillation of the entire system without stretching the springs. For the two degenerate mode vectors, we check whether $\overrightarrow{\Phi}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\overrightarrow{\Phi}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ satisfy the normalization condition. One has

$$\begin{aligned} &\overrightarrow{\Phi}_2^T \mathbf{t} \overrightarrow{\Phi}_2 = \overrightarrow{\Phi}_3^T \mathbf{t} \overrightarrow{\Phi}_3 = 1 \\ &\overrightarrow{\Phi}_1^T \mathbf{t} \overrightarrow{\Phi}_2 = \overrightarrow{\Phi}_1^T \mathbf{t} \overrightarrow{\Phi}_3 = 0 \\ &\overrightarrow{\Phi}_2^T \mathbf{t} \overrightarrow{\Phi}_3 = 0 \end{aligned}$$

So $\overrightarrow{\Phi}_2$ and $\overrightarrow{\Phi}_3$ are indeed the $\omega_1^2 = 1 + \epsilon$ mode vectors. In mode $\overrightarrow{\Phi}_2$, the pendulum in the middle doesn't oscillate while the other two oscillate with phase difference π. In mode $\overrightarrow{\Phi}_3$, the outer pendula oscillate in phase while the one in the middle oscillates with phase difference π.

Problem 5

This is a *very* challenging problem, but it is also an excellent example of finding an equilibrium and expanding around it when it's not just a matter of blindly Taylor expanding. We note that our first solution, while largely correct (modulo a missing factor of 1/2 due to algebraic errors), was not complete because it assume E in the relativistic term was constant. The solution should have checked the veracity fo this assumption by recalculating the energy at the end (including the oscillation energy) and showing that it was valid to take E to be constant to the level of approximation we used.

However, in thinking about that, we discovered a cleaner way to solve the problem that doesn't require this self-consistency check. It's also much closer to the way equilibria are discussed in the notes in terms of derivatives of the potential, so is probably closer to the language you are used to.

Let's first look at the problem conceptually. There are thrre things we need to do:

1. Find the new equilibrium value of the circular orbit, r_0, which will differ from the nonrelativistic case because of the relativistic correction to the potential.

2. Expand about this new r_0 to find the frequency of radial oscillations.

3. Compare the frequency of radial oscillation to the orbital period for the circular orbit at the new relativistic radius to determine the angular velocity of precession.

In terms of the mechanics of the expansion, one thing we run into very early is the appearance of a kinetic energy of radial oscillation in the relativistic correction to the potential. How do we deal with that? We recognize that this term looks like a correction to the radial oscillation kinetic energy term. Rather than having to say by fiat that this term ought to be moved to the kinetic energy, we can let it happen naturally by expanding not the potential only but rather by expanding the full Lagrangian. This is the technique that is presented in Section 3.1.1 of the lecture notes, so is not foreign to you (but likely has been forgotten). We will see the solution evolve naturally when we do this.

We begin with the effective 1-dimensional Lagrangian for the central force problem:

$$L_{1D}(r,\dot{r}) = \frac{1}{2}\,\mu\,\dot{r}^2 - \frac{l^2}{2\,\mu\,r^2} - U(r) \tag{1}$$

Note the negative sign on the centrifugal term! Recall that this term receives a sign flip when going from the 3D Lagrangian (where it is the θ kinetic energy and necessarily has a positive sign) to the effective 1D Lagrangian. This was highlighted in the notes (Section 4.1.1). We now insert the form for $U(r)$ provided:

$$L_{1D}(r,\dot{r}) = \frac{1}{2}\,\mu\,\dot{r}^2 - \frac{l^2}{2\,\mu\,r^2} - V(r) + \frac{1}{2\,\mu\,c^2}\left[E_{NR} - V(r)\right]^2 \tag{2}$$

where we explicitly include a $_{NR}$ subscript on E in the correction term to indicate it is the nonrelativistic energy. Let's write out $E_{NR} - V(r)$:

$$E_{NR} - V(r) = \left[\frac{l^2}{2\,\mu\,r^2} + \frac{1}{2}\,\mu\,\dot{r}^2 + V(r)\right] - V(r) = \left[\frac{l^2}{2\,\mu\,r^2} + \frac{1}{2}\,\mu\,\dot{r}^2\right] \tag{3}$$

So far, we have just followed our nose – it is straightforward to write down the total nonrelativistic energy in a central force problem. Here, of course, the l^2 term has a positive sign because we are considering the energy, not a Lagrangian. Let's insert this into 1D Lagrangian and collect terms:

$$L_{1D}(r,\dot{r}) = \frac{1}{2}\,\mu\,\dot{r}^2 - \frac{l^2}{2\,\mu\,r^2} - V(r) + \frac{1}{2\,\mu\,c^2}\left[\left(\frac{l^2}{2\,\mu\,r^2}\right)^2 + \left(\frac{1}{2}\,\mu\,\dot{r}^2\right)^2 + 2\left(\frac{l^2}{2\,\mu\,r^2}\right)\left(\frac{1}{2}\,\mu\,\dot{r}^2\right)\right]$$
$$= \frac{1}{2}\,\mu\,\dot{r}^2\left[1 + \frac{l^2}{2\,\mu^2\,r^2\,c^2} + \frac{1}{4}\,\frac{\dot{r}^2}{c^2}\right] - \frac{l^2}{2\,\mu\,r^2}\left[1 - \frac{1}{2}\,\frac{l^2}{2\,\mu^2\,r^2\,c^2}\right] - V(r) \tag{4}$$

There are two correction terms to the radial kinetic energy, one of the form $l^2/\mu^2 r^2 c^2 \approx v_\theta^2/c^2$ because $l \approx \mu\,v_\theta\,r$ for nearly circular orbits and the other of the form v_r^2/c^2. The correction term to the centrifugal term is also of the form $l^2/\mu^2 r^2 c^2 \approx v_\theta^2/c^2$.

So far we have made no approximations. To continue from here, though, we need to assume the orbit is close to circular. We shall write $r = r_0 + q$ where r_0 is the exact relativistic circular orbit and q is a small perturbation away from r_0. The radial velocity is now $\dot{r} = \dot{q}$ because $\dot{r} = 0$ for the circular orbit. We now must Taylor expand everything in sight about $r = r_0$ and $\dot{r} = 0$:

$$\begin{aligned}
L_{1D}\left(r_0 + q, \dot{q}\right) \approx{}& \left(\frac{1}{2}\,\mu\,\dot{r}^2\Big|_{\dot{r}_0=0} + \mu\,\dot{r}|_{\dot{r}_0=0}\,\dot{q} + \frac{1}{2}\,\mu|_{\dot{r}=0}\,\dot{q}^2\right) \\
&\times\left[1 + \left(\frac{l^2}{2\,\mu^2 r_0^2\,c^2} - 2\,\frac{l^2}{2\,\mu^2 r_0^3\,c^2}\,q + \frac{6}{2}\,\frac{l^2}{2\,\mu^2 r_0^4\,c^2}\,q^2\right)\right. \\
&\qquad\left. + \frac{1}{2\,\mu\,c^2}\left(\frac{1}{2}\,\mu\,\dot{r}^2\Big|_{\dot{r}=0} + \mu\,\dot{r}|_{\dot{r}_0=0}\,\dot{q} + \frac{1}{2}\,\mu|_{\dot{r}_0=0}\,\dot{q}^2\right)\right] \\
&- \left(\frac{l^2}{2\,\mu\,r_0^2} - 2\,\frac{l^2}{2\,\mu\,r_0^3}\,q + \frac{6}{2}\,\frac{l^2}{2\,\mu\,r_0^4}\,q^2\right) \\
&\times\left[1 - \frac{1}{2}\left(\frac{l^2}{2\,\mu^2 r_0^2\,c^2} - 2\,\frac{l^2}{2\,\mu^2 r_0^3\,c^2}\,q + \frac{6}{2}\,\frac{l^2}{2\,\mu^2 r_0^4\,c^2}\,q^2\right)\right] \\
&- V(r_0) - \frac{dV}{dr}\Big|_{r_0} q - \frac{1}{2}\,\frac{d^2V}{dr^2}\Big|_{r_0} q^2 \\
={}& \frac{1}{2}\,\mu\,\dot{q}^2\left[1 + \frac{1}{2}\,\frac{l^2}{\mu^2 r_0^2\,c^2} - \frac{l^2}{\mu^2 r_0^3\,c^2}\,q + \frac{3}{2}\,\frac{l^2}{\mu^2 r_0^4\,c^2}\,q^2 + \frac{1}{4}\,\frac{\dot{q}^2}{c^2}\right] \\
&- \left(\frac{1}{2}\,\frac{l^2}{\mu\,r_0^2} - \frac{l^2}{\mu\,r_0^3}\,q + \frac{3}{2}\,\frac{l^2}{\mu\,r_0^4}\,q^2\right) \\
&\times\left[1 - \frac{1}{4}\frac{l^2}{\mu^2\,r_0^2\,c^2} + \frac{1}{2}\,\frac{l^2}{\mu^2\,r_0^3\,c^2}\,q - \frac{3}{4}\,\frac{l^2}{\mu^2\,r_0^4\,c^2}\,q^2\right] \\
&- V(r_0) - \frac{dV}{dr}\Big|_{r_0} q - \frac{1}{2}\,\frac{d^2V}{dr^2}\Big|_{r_0} q^2
\end{aligned}$$

We have been painfully explicit about the expansion of the kinetic energy term because all the zeroth- and first-order terms vanish because the value of the term and its first derivative at the equilibrium point vanish because $\dot{r} = 0$ there.

Now, we will start discarding terms that are of too high order. We are really considering two orders of expansion right now. There is the relativistic expansion, which comes with terms like $\dot{q}^2/c^2$ and $l^2/\mu^2\,r_0^2\,c^2 \approx v_\theta^2/c^2$, and there is the spatial Taylor expansion, which yields terms in q and q^2. It is certainly true that $\dot{q}/v_\theta \ll 1$. So will neglect anything with the factor $\dot{q}^2/c^2$: these are second-order relativistic terms. In q, though, we need to go up to order q^2 because the q term's coefficient will be required to vanish in order for $r = r_0$ to be an equilibrium. Any terms of order

$\dot{q}^2 q$ or $\dot{q}^2 q^2$ can be discarded because the both $\dot{q}$ and q are small, so these terms are third or fourth order in the spatial expansion. So we have

$$
\begin{aligned}
L_{1D}(r_0 + q, \dot{q}) = & \frac{1}{2}\,\mu\,\dot{q}^2 \left(1 + \frac{l^2}{2\,\mu^2 r_0^2\, c^2}\right) \\
& - \left(\frac{1}{2}\,\frac{l^2}{\mu\, r_0^2} - \frac{l^2}{\mu\, r_0^3}\, q + \frac{3}{2}\,\frac{l^2}{\mu\, r_0^4}\, q^2\right)\left(1 - \frac{1}{4}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) \\
& - \left(\frac{1}{2}\,\frac{l^2}{\mu\, r_0^2} - \frac{l^2}{\mu\, r_0^3}\, q\right)\left(\frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^3\, c^2}\, q\right) + \left(\frac{1}{2}\,\frac{l^2}{\mu\, r_0^2}\right)\left(\frac{3}{4}\,\frac{l^2}{\mu^2\, r_0^4\, c^2}\, q^2\right) \\
& - V(r_0) - \left.\frac{dV}{dr}\right|_{r_0} q - \frac{1}{2}\left.\frac{d^2V}{dr^2}\right|_{r_0} q^2
\end{aligned}
$$

Yes, there are lots of terms, but we'll see it cleans up. Let's group by order in q and relativistic expansion:

$$
\begin{aligned}
L_{1D}(r_0 + q, \dot{q}) \approx & -\left(\frac{l^2}{2\,\mu\, r_0^2}\right)\left(1 - \frac{1}{4}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - V(r_0) \\
& + \frac{1}{2}\,\mu\,\dot{q}^2 \left(1 + \frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) \\
& + q\left[\frac{l^2}{\mu\, r_0^3}\left(1 - \frac{1}{4}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - \left(\frac{1}{2}\,\frac{l^2}{\mu\, r_0^2}\right)\left(\frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^3\, c^2}\right) - \left.\frac{dV}{dr}\right|_{r_0}\right] \\
& + q^2\left[-\frac{3}{2}\,\frac{l^2}{\mu\, r_0^4} + \frac{3}{8}\,\frac{l^4}{\mu^3\, r_0^6\, c^2} + \frac{1}{2}\,\frac{l^4}{\mu^3\, r_0^6\, c^2} + \frac{3}{8}\,\frac{l^4}{\mu^3\, r_0^6\, c^2} - \frac{1}{2}\left.\frac{d^2V}{dr^2}\right|_{r_0}\right] \\
= & -\left(\frac{l^2}{2\,\mu\, r_0^2}\right)\left(1 - \frac{1}{4}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - V(r_0) \\
& + q\left[\frac{l^2}{\mu\, r_0^3}\left(1 - \frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - \left.\frac{dV}{dr}\right|_{r_0}\right] \\
& + \frac{1}{2}\,\mu\,\dot{q}^2 \left(1 + \frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) + q^2\left[\left(\frac{l^2}{\mu\, r_0^4}\right)\left(-\frac{3}{2} + \frac{5}{4}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - \frac{1}{2}\left.\frac{d^2V}{dr^2}\right|_{r_0}\right]
\end{aligned}
$$

With the terms grouped in this way, we see: 1) a constant term that we can ignore; 2) a term linear in q who coefficient must vanish in order for r_0 to be an equilibrium; 3) a radial kinetic energy term with essentially a relativistic correction to the reduced mass; and 4) a term quadratic in q that gives the restoring force that drives the oscillation in radius around r_0. We first work on (3) to find r_0:

$$
\frac{l^2}{\mu\, r_0^3}\left(1 - \frac{l^2}{2\,\mu^2\, r_0^2\, c^2}\right) - \left.\frac{dV}{dr}\right|_{r_0} = 0 \tag{5}
$$

At this point, we need to use the actual form of the nonrelativistic force law, which yields

$$\begin{aligned}
\frac{l^2}{\mu\, r_0^3}\left(1-\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)-\frac{G\,\mu\,M}{r_0^2}&=0\\
\frac{l^2}{G\,\mu^2\,M}\left(1-\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)-r_0&=0\\
r_0=r_{0,NR}\left(1-\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)&\approx r_{0,NR}\left(1-\frac{l^2}{2\,\mu^2\,r_{0,NR}^2\,c^2}\right)
\end{aligned} \tag{6}$$

where $r_{0,NR}=l^2/G\,\mu^2\,M$ is the scale radius for the nonrelativistic case, which is just the radius of the circular orbit in the nonrelativistic case. We have replaced r_0 with $r_{0,NR}$ in the relativistic correction factor because including the error that results is of the next highest order in relativistic correction. We thus have the relativistic circular orbit radius in terms of the nonrelativistic circular orbit radius and a correction factor that deviates from unity by a relativistic term. We note that $r_0 < r_{0,NR}$. This makes sense given Equation 4: in the initial 1-dimensional Lagrangian, the kinetic energy gains terms that essentially increase the effective mass, and the centrifugal term is reduced in magnitude by the relativistic correction. Both those effects – increasing the mass and decreasing the centrifugal repulsion – will tend to make the radius of the orbit smaller.

Though it is unimportant, we note that we may reduce the constant term by explicitly writing $V(r_0)$:

$$\begin{aligned}
\text{constant term}&=-\left(\frac{l^2}{2\,\mu\,r_0^2}\right)\left(1-\frac{1}{2}\,\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)-V(r_0)\\
&=-\left(\frac{l^2}{2\,\mu\,r_0^2}\right)\left(1-\frac{1}{2}\,\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)+\frac{G\,\mu\,M}{r_0}\\
&=\left(\frac{l^2}{\mu\,r_0^2}\right)\left(-\frac{1}{2}+\frac{1}{8}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}+\frac{r_0}{r_0^{NR}}\right)\\
&=\left(\frac{l^2}{\mu\,r_0^2}\right)\left(-\frac{1}{2}+\frac{1}{8}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}+1-\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)\\
&=\left(\frac{l^2}{2\,\mu\,r_0^2}\right)\left(1-\frac{3}{4}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)
\end{aligned}$$

Of course, we want to rewrite everything in terms of nonrelativistic quantities, so let's use our expression for r_0 in terms of $r_{0,NR}$:

$$\begin{aligned}
\text{constant term}&=\left(\frac{l^2}{2\,\mu\,r_{0,NR}^2}\right)\left(1-\frac{1}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)^{-2}\left(1-\frac{3}{4}\,\frac{l^2}{\mu^2\,r_{0,NR}^2\,c^2}\right)\\
&=\left(\frac{l^2}{2\,\mu\,r_{0,NR}^2}\right)\left(1+\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\left(1-\frac{3}{4}\,\frac{l^2}{\mu^2\,r_{0,NR}^2\,c^2}\right)\\
&=\left(\frac{l^2}{2\,\mu\,r_{0,NR}^2}\right)\left(1+\frac{1}{4}\,\frac{l^2}{\mu^2\,r_{0,NR}^2\,c^2}\right)
\end{aligned} \tag{7}$$

where in the first step we have used the expression for $r_0/r_{0,NR}$ for the $1/r_0^2$ in the prefactor, but just replace r_0 with $r_{0,NR}$ in the correction term because that correction term is already small. The result is just the expected nonrelativistic expression $l^2/2\,\mu\,r_{0,NR}^2$ with a relativistic correction.

We need to write out more clearly the quadratic term to obtain the frequency of radial oscillation about r_0. We write out the d^2V/dr^2 term and clean up:

$$\begin{aligned}
\text{quadratic term} &= q^2\left[\left(\frac{l^2}{\mu\,r_0^4}\right)\left(-\frac{3}{2}+\frac{5}{4}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)-\frac{1}{2}\,\frac{d^2V}{dr^2}\bigg|_{r_0}\right]\\
&= q^2\left[\left(\frac{l^2}{\mu\,r_0^4}\right)\left(-\frac{3}{2}+\frac{5}{4}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)-\frac{1}{2}\left(-2\,\frac{G\,\mu\,M}{r_0^3}\right)\right]\\
&= q^2\left(\frac{l^2}{\mu\,r_0^4}\right)\left[\left(-\frac{3}{2}+\frac{5}{4}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)+\frac{r_0}{r_{0,NR}}\right]
\end{aligned}$$

where we again have used $r_{0,NR} = l^2/G\,\mu^2\,M$. We may substitute in using our expression above for r_0 in terms of $r_{0,NR}$:

$$\begin{aligned}
\text{quadratic term} &= q^2\left(\frac{l^2}{\mu\,r_0^4}\right)\left[\left(-\frac{3}{2}+\frac{5}{4}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)+\left(1-\frac{l^2}{2\,\mu^2 r_0^2 c^2}\right)\right]\\
&= -\frac{1}{2}\,\mu\,q^2\left(\frac{l^2}{\mu^2\,r_0^4}\right)\left(1-\frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)
\end{aligned}$$

where we use the more exact expression since the other relativistic correction terms are in similar form.

We thus have for the 1-dimensional Lagrangian, after discarding the constant term and the vanishing linear term and including the simplified quadratic term::

$$L_{1D}(r_0+q,\dot{q}) \approx \frac{1}{2}\,\mu\,\dot{q}^2\left(1+\frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)-\frac{1}{2}\,\mu\,q^2\left(\frac{l^2}{\mu^2\,r_0^4}\right)\left(1-\frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)$$

This is a simple harmonic oscillator Lagrangian. Recalling from the lecture notes that the square of the natural frequency is the magnitude of the ratio of the coefficients of the q^2 and $\dot{q}^2$ terms, we have

$$\begin{aligned}
\omega_{osc}^2 &= \left(\frac{l^2}{\mu^2\,r_0^4}\right)\left(1-\frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\left(1+\frac{1}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)^{-1}\\
&= \left(\frac{v_\theta}{r_0}\right)^2\left(1-2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\\
&= \left(\frac{2\,\pi}{\tau}\right)^2\left(1-2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)
\end{aligned}$$

where we have used $2\,\pi\,r_0/v_\theta = \tau$ where τ is the period of the relativistic circular orbit. So the radial oscillation period is

$$\tau_{osc} = \frac{2\,\pi}{\omega_{osc}} = \tau\left(1-2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)^{-1/2} = \tau\left(1+\frac{l^2}{\mu^2\,r_0^2\,c^2}\right) \qquad (8)$$

For completeness, we also would like to relate τ and τ_{NR}, where the latter is the period of a nonrelativistic circular orbit. One must take a bit of care in understanding the correspondence between relativistic and nonrelativistic orbits. Since l completely determines the orbit for circular orbits (since E is determined by l in this case), let us consider relativistic and nonrelativistic orbits with the same l. This is consistent with how we calculated r_0 in terms of $r_{0,NR}$ before: we took $r_{0,NR}$

to be the nonrelativistic orbit with the same l as the relativistic orbit. We need v_θ to determine the orbital period. It certainly holds that

$$\mu\, r_{0,NR}\, v_{\theta,NR} = l = \mu\, r_0\, v_\theta$$

Rearranging, we thus have

$$\begin{aligned} v_\theta = v_{\theta,NR}\,\frac{r_{0,NR}}{r_0} &= v_{\theta,NR}\left(1 - \frac{1}{2}\,\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right)^{-1} \\ &= v_{\theta,NR}\left(1 + \frac{1}{2}\,\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right) \end{aligned} \tag{9}$$

which is as you would expect – if $r_0 < r_{0,NR}$ for the same l, then $v_\theta > v_{\theta,NR}$ in order for the l values to match. Using this now to calculate the periods:

$$\begin{aligned} \tau = \frac{2\,\pi\, r_0}{v_\theta} &= \left(\frac{r_0}{r_{0,NR}}\right)\left(\frac{v_{\theta,NR}}{v_\theta}\right)\frac{2\,\pi\, r_{0,NR}}{v_{\theta,NR}} \\ &= \left(1 - \frac{1}{2}\,\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right)\left(1 + \frac{1}{2}\,\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right)^{-1}\tau_{NR} \\ &= \left(1 - \frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right)\tau_{NR} \end{aligned} \tag{10}$$

The angular precession speed is determined by the ratio τ_{osc}/τ because this tells us what fraction of an orbit is required to complete one radial oscillation. The angular speed of precession is given by the change in the angle of the apside per orbital period, where the apside is the point at which the radial oscillation returns to 0. Therefore, the precession speed is

$$\begin{aligned} \Omega_p &= \frac{2\,\pi}{\tau}\left(\frac{\tau_{osc}}{\tau} - 1\right) = 2\,\pi\left(\frac{v_\theta}{2\,\pi\, r_0}\right)\left(\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right) = \left(\frac{l}{\mu\, r_0^2}\right)\left(\frac{l^2}{\mu^2\, r_{0,NR}^2\, c^2}\right) \\ &= \frac{l^3}{\mu^3\, r_0^4\, c^2} \end{aligned} \tag{11}$$

The above result can be rewritten in a couple of interesting forms. If we replace l using $l^2 = G\,\mu^2\, M\, r_{0,NR} \approx G\,\mu^2\, M\, r_0$, we have

$$\Omega_p = \frac{\left(G\,\mu^2\, M\, r_0\right)^{3/2}}{\mu^3\, r_0^4\, c^2} = \frac{(G\, M)^{3/2}}{r_0^{5/2}\, c^2} \tag{12}$$

The same expression could be rewritten with $r_{0,NR}$ instead and be just as true – the difference between them would be a higher-order relativistic correction. If we instead use $l = \mu\, r_0\, v_\theta$, we have

$$\Omega_p = \frac{v_\theta}{r_0}\,\frac{v_\theta^2}{c^2} = \frac{2\,\pi}{\tau}\,\frac{v_\theta^2}{c^2} \tag{13}$$

which is a nice way of seeing that Ω_p is a purely relativistic quantity. Again, τ and v_θ could be replaced with their nonrelativistic version and the expression would be as correct.

An alternate method for deriving the precession frequency is to calculate a force using the 1-dimensional Lagrangian and then put that into the orbit differential equation, as was suggested by the hint given in the problem set. Recall that the orbit differential equation is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu\, r^2}{l^2}\, F(r)$$

where the u term arises from the centrifugal term in the 1D potential. Returning to the 1D Lagrangian, Equation 4, we now know which terms survive the Taylor expansion, so we know how we need to modify the orbit differential equation to include those terms. First, we must add a term to the force law for the additional potential term

$$-\frac{1}{2}\,\frac{l^2}{2\,\mu\, r^2}\,\frac{l^2}{2\,\mu^2\, r^2\, c^2}$$

(It appears in L_{1D} with a + sign, but recall that V appears with a negative sign in the Lagrangian). Second, we must multiply the kinetic energy term by

$$1 + \frac{l^2}{2\,\mu^2\, r^2\, c^2}$$

because this correction factor multipled the kinetic energy term in L_{1D}. The only term we are not including is the $\dot{r}^2/c^2$ correction to the kinetic energy term, which we dropped before because it yielded second-order relativistic corrections.

So the orbit differential equation becomes

$$\left(1 + \frac{l^2}{2\,\mu^2\, r^2\, c^2}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu\, r^2}{l^2}\left[-\frac{dV}{dr} - \frac{d}{dr}\left(-\frac{1}{2}\,\frac{l^2}{2\,\mu\, r^2}\,\frac{l^2}{2\,\mu^2\, r^2\, c^2}\right)\right]$$
$$\left(1 + \frac{l^2}{2\,\mu^2\, r^2\, c^2}\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu\, r^2}{l^2}\left(-\frac{dV}{dr} - \frac{1}{2}\,\frac{l^4}{\mu^3\, r^5\, c^2}\right)$$

Again, we presume there to be a circular orbit at $r = r_0$ and Taylor-expand the above equation about r_0 with $r = r_0 + q$. This time, though, we only need to expand to first order in q because we are writing the force equation, not an energy equation.

$$\left(1 + \frac{l^2}{2\,\mu^2\, r_0^2\, c^2} - \frac{l^2}{\mu^2\, r_0^3\, c^2}\, q\right)\frac{d^2}{d\theta^2}\left(\frac{1}{r_0} - \frac{q}{r_0^2}\right) + \left(\frac{1}{r_0} - \frac{q}{r_0^2}\right)$$
$$= \frac{\mu}{l^2}\left(r_0^2 + 2\, r_0\, q\right)\left(\left.\frac{dV}{dr}\right|_{r_0} + \left.\frac{d^2V}{dr^2}\right|_{r_0} q + \frac{1}{2}\,\frac{l^4}{\mu^3\, r_0^5\, c^2} - \frac{5}{2}\,\frac{l^4}{\mu^3\, r_0^6\, c^2}\, q\right)$$
$$-\frac{1}{r_0^2}\left(1 + \frac{l^2}{2\,\mu^2\, r_0^2\, c^2}\right)\frac{d^2 q}{d\theta^2} + \left(\frac{1}{r_0} - \frac{q}{r_0^2}\right)$$
$$= \frac{\mu}{l^2}\left[r_0^2\left.\frac{dV}{dr}\right|_{r_0} + \frac{1}{2}\,\frac{l^4}{\mu^3\, r_0^3\, c^2} + q\left(2\, r_0\left.\frac{dV}{dr}\right|_{r_0} + \frac{l^4}{\mu^3\, r_0^4\, c^2} + r_0^2\left.\frac{d^2V}{dr^2}\right|_{r_0} - \frac{5}{2}\,\frac{l^4}{\mu^3\, r_0^4\, c^2}\right)\right]$$

The equation must obviously hold at $q = 0$, which gives us a condition that allows us to determine r_0 as well as eliminate the constant terms from both sides. First, the condition is

$$\frac{1}{r_0} = \frac{\mu}{l^2}\left(r_0^2\left.\frac{dV}{dr}\right|_{r_0} + \frac{1}{2}\,\frac{l^4}{\mu^3\, r_0^3\, c^2}\right)$$
$$\frac{l^2}{\mu\, r_0^3}\left(1 - \frac{1}{2}\,\frac{l^2}{\mu^2\, r_0^2\, c^2}\right) - \left.\frac{dV}{dr}\right|_{r_0} = 0$$

which we see is the same as Equation 5. Hence the result for r_0 will be identical to that obtained before

$$r_0 = r_{0,NR}\left(1 - \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right) \approx r_{0,NR}\left(1 - \frac{l^2}{2\,\mu^2\,r_{0,NR}^2\,c^2}\right)$$

Rewriting the orbit differential equation using the relation found by requiring it to hold at $q = 0$ gives

$$\begin{aligned}
-\frac{1}{r_0^2}\left(1 + \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)\frac{d^2q}{d\theta^2} - \frac{q}{r_0^2} &= \frac{\mu\,q}{l^2}\left(2\,r_0\,\left.\frac{dV}{dr}\right|_{r_0} + r_0^2\,\left.\frac{d^2V}{dr^2}\right|_{r_0} - \frac{3}{2}\,\frac{l^4}{\mu^3\,r_0^4\,c^2}\right) \\
\frac{d^2q}{d\theta^2} &= -\,q\,r_0^2\left(\frac{1}{r_0^2} + \frac{2\,\mu\,r_0}{l^2}\,\left.\frac{dV}{dr}\right|_{r_0} + \frac{\mu\,r_0^2}{l^2}\,\left.\frac{d^2V}{dr^2}\right|_{r_0} - \frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^4\,c^2}\right)\left(1 + \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right)^{-1}
\end{aligned}$$

Now we must use the explicit form of V:

$$\begin{aligned}
\frac{d^2q}{d\theta^2} &= -\,q\left(1 + \frac{2\,\mu\,r_0^3}{l^2}\left(\frac{G\,\mu\,M}{r_0^2}\right) + \frac{\mu\,r_0^4}{l^2}\left(-\frac{2\,G\,\mu\,M}{r_0^3}\right) - \frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\left(1 - \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right) \\
\frac{d^2q}{d\theta^2} &= -\,q\left(1 + 2\,\frac{G\,\mu^2\,M}{l^2}\,r_0 - 2\,\frac{G\,\mu^2\,M}{l^2}\,r_0 - \frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\left(1 - \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right) \\
\frac{d^2q}{d\theta^2} &= -\,q\left(1 - \frac{3}{2}\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)\left(1 - \frac{l^2}{2\,\mu^2\,r_0^2\,c^2}\right) \\
\frac{d^2q}{d\theta^2} &= -\,q\left(1 - 2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right) \qquad \text{or, equivalently} \qquad \frac{d^2q}{d\theta^2} + q\left(1 - 2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right) = 0
\end{aligned}$$

We have obtained a harmonic oscillator equation in $q(\theta)$. Since this is an equation with θ as the independent variable, not t, the frequency of oscillation is in units of the orbital angular frequency. That is:

$$\left(1 - 2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right) \equiv \omega_{osc,\theta}^2 = \left(\frac{\omega_{osc}}{2\,\pi/\tau}\right)^2$$

where $\tau = 2\,\pi\,r_0/v_\theta$ is the orbital period. Fortunately, since we were interested mainly in τ_{osc}/τ anyways, this ratio is what we want:

$$\frac{\tau_{osc}}{\tau} = \omega_{osc,\theta}^{-1} = \left(1 - 2\,\frac{l^2}{\mu^2\,r_0^2\,c^2}\right)^{-1/2} = \left(1 + \frac{l^2}{\mu^2\,r_0^2\,c^2}\right)$$

which we see is the same result as we obtained earlier for τ_{osc}/τ. Therefore, the precession frequency result will be the same,

$$\Omega_p = \frac{(G\,M)^{3/2}}{r_0^{5/2}\,c^2} = \frac{2\,\pi}{\tau}\,\frac{v_\theta^2}{c^2} \tag{14}$$

Problem 6

Let θ^* be the scattering angle in the center of mass frame, and θ be the scattering angle in the lab frame. From the lecture notes, one has

$$\tan\theta = \frac{\sin\theta^*}{\frac{m_1}{m_2} + \cos\theta^*} = \frac{\sin\theta^*}{\gamma + \cos\theta^*} \approx \frac{\sin\theta^*}{\gamma}$$

where we use fact $\gamma \gg 1$. Since $\tan\theta \ll 1$, we get from the above equation

$$\sin\theta^* = \gamma\tan\theta \approx \gamma\theta$$

where we notice

$$\gamma\theta < 1$$

for $\sin\theta^* < 1$.

We may obtain the solution by simply using the center-of-mass frame Rutherford cross section and changing variables to the lab frame. The CM frame differential cross section is

$$\frac{d\sigma}{d\Omega_*} = \left(\frac{1}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta_*}{2}}$$

We use $E = \frac{1}{2}m_2 v_0^2$ because that is the energy of the incoming particle in the CM frame. We may use trigonometric identities to evaluate $\sin^4\frac{\theta_*}{2}$ in terms of γ and θ:

$$\sin^2\frac{\theta_*}{2} = \frac{1}{2}\left(1-\cos\theta_*\right) = \frac{1}{2}\left(1-\sqrt{1-\sin^2\theta_*}\right) \approx \frac{1}{2}\left(1-\sqrt{1-\gamma^2\theta^2}\right)$$

The other thing we need to do is calculate the Jacobian transformation of $\frac{d\sigma}{d\Omega}$. $d\sigma$ is unchanged because it refers to a quantity measured in a plane transverse to the direction of the velocity of the CM frame. $d\Omega_*$ on the other hand satisfies:

$$d\Omega_* = 2\,\pi\,\sin\theta_* d\theta_* = 2\,\pi\,d\left(\cos\theta_*\right) = 2\,\pi\,d\left(\sqrt{1-\gamma^2\theta^2}\right) = 2\,\pi\,\frac{\gamma^2\theta\,d\theta}{\sqrt{1-\gamma^2\theta^2}} = \frac{\gamma^2 d\Omega}{\sqrt{1-\gamma^2\theta^2}}$$

Putting it all together, we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{\gamma^2}{\sqrt{1-\gamma^2\theta^2}}\frac{d\sigma}{d\Omega_*} \\
&= \frac{\gamma^2}{\sqrt{1-\gamma^2\theta^2}}\left(\frac{1}{2m_2v_0^2}\right)^2 \frac{1}{\left[\frac{1}{2}\left(1-\sqrt{1-\gamma^2\theta^2}\right)\right]^2} \\
&= \left(\frac{1}{2m_2v_0^2}\right)^2 \frac{4\gamma^2}{\sqrt{1-\gamma^2\theta^2}\left[\left(1-\sqrt{1-\gamma^2\theta^2}\right)\right]^2}
\end{aligned}$$

Since no particle can go to $\sin\theta_* > 1$, *i.e.*, $\gamma\theta > 1$,

$$\frac{d\sigma}{d\Omega} = 0 \qquad \text{for} \qquad \gamma\theta > 1$$

Problem 7

No students have turned in Problem 7 yet, so we will delay posting the solution.

Problem 8

Since the total angular momentum is conserved, we can assume it is along z-axis. So conservation of the angular momentum gives us three constraints

$$G_1 = \sum_k l_k \cos\theta_k = L_0 = \left|\overrightarrow{L}\right|$$
$$G_2 = \sum_k l_k \cos\phi_k \sin\theta_k = 0$$
$$G_3 = \sum_k l_k \sin\phi_k \sin\theta_k = 0$$

where $\overrightarrow{l}_k$ is the angular momentum of the k^{th} particle. The total energy is

$$E_T = \sum_k E_k = \sum_k \frac{m_k}{2}\left(\frac{Gm_kM}{l_k}\right)^2\left(\epsilon_k^2 - 1\right) = \sum_k \frac{\alpha_k}{l_k^2}\left(\epsilon_k^2 - 1\right)$$

where $\alpha_k = \frac{m_k}{2}\left(Gm_kM\right)^2$, ϵ_k is the eccentricity of the orbit of the k^{th} particle and $\mu_k = m_k$ since $m_k \ll M$. Minimizing E_T subject to these constraints G_1, G_2 ,and G_3 gives us

$$\frac{\partial E_T}{\partial q_k} + \sum_i \lambda_i \frac{\partial G_i}{\partial q_k} = 0$$

where q_k are independent variables of k^{th} particle, such as $\theta_k, \phi_k, \epsilon_k$ and l_k. So we have

$$\frac{\partial E_T}{\partial \epsilon_k} + \sum_i \lambda_i \frac{\partial G_i}{\partial \epsilon_k} = 0 \Rightarrow \frac{2\alpha_k}{l_k^2}\epsilon_k = 0 \Rightarrow \epsilon_k = 0$$

which means all particles travel on circular orbits. While we have

$$\frac{\partial E_T}{\partial \phi_k} + \sum_i \lambda_i \frac{\partial G_i}{\partial \phi_k} = 0$$
$$\Rightarrow -\lambda_2 \sin\theta_k \sin\phi_k + \lambda_3 \sin\theta_k \cos\phi_k = 0$$
$$\Rightarrow \frac{\lambda_3}{\lambda_2} = \tan\phi_k \text{ or } \sin\theta_k = 0$$

If $\frac{\lambda_3}{\lambda_2} = \tan\phi_k$, one has

$$\frac{\partial E_T}{\partial \theta_k} + \sum_i \lambda_i \frac{\partial G_i}{\partial \theta_k} = 0$$
$$\Rightarrow -\lambda_1 \sin\theta_k + \lambda_2 \cos\theta_k \cos\phi_k + \lambda_3 \cos\theta_k \sin\phi_k = 0$$
$$\Rightarrow \frac{\lambda_2}{\lambda_1} = \cos\phi_k \tan\theta_k$$

However, G_2 and G_3 yield

$$G_2 = \sum_k l_k \cos\phi_k \sin\theta_k = \frac{\lambda_2}{\lambda_1}\sum_k l_k \cos\theta_k = \frac{\lambda_2}{\lambda_1}L_0 = 0$$
$$G_3 = \sum_k l_k \sin\phi_k \sin\theta_k = \frac{\lambda_3}{\lambda_1}\sum_k l_k \cos\theta_k = \frac{\lambda_3}{\lambda_1}L_0 = 0$$

Since $L_0 \neq 0$, it therefore holds

$$\lambda_2 = \lambda_3 = 0$$

which imply

$$\frac{\lambda_2}{\lambda_1} = \cos\phi_k \tan\theta_k = 0$$
$$\frac{\lambda_3}{\lambda_1} = \sin\phi_k \tan\theta_k = 0$$

and therefore we still come to $\theta_k = 0$.At last one also has

$$\frac{\partial E_T}{\partial l_k} + \sum_i \lambda_i \frac{\partial G_i}{\partial l_k} = 0$$
$$\Rightarrow \frac{2\alpha_k}{l_k^3} + \lambda_1 \cos\theta_k + \lambda_2 \cos\phi_k \sin\theta_k + \lambda_3 \sin\phi_k \sin\theta_k = 0$$
$$\Rightarrow r_k^{\frac{3}{2}} \propto \frac{m_k^3}{l_k^3} \propto \frac{\alpha_k}{l_k^3} = -\frac{\lambda_1}{2}$$

So r_k doesn't depend on k and thereby the material must all lie on a circular ring. For k^{th} particle, one has

$$r_k = \frac{l_k^2}{GMm_k^2} \Rightarrow l_k = m_k\sqrt{GMr_k}$$

and G_1 gives us

$$\sum_k m_k\sqrt{GMr_k} = L_0$$
$$\Rightarrow \sqrt{GMr}\sum_k m_k = L_0$$
$$\Rightarrow r = \frac{1}{GM}\left(\frac{L_0}{m}\right)^2$$

where we use $\sum_k m_k = m$. And we also find

$$l_k = \frac{m_k}{m}L_0$$

which means

$$E_T = -\sum_k \frac{m_k}{2}\left(\frac{Gm_kM}{l_k}\right)^2 = -\sum_k \frac{m_k}{2}\left(\frac{GmM}{L_0}\right)^2 = -\frac{m}{2}\left(\frac{GmM}{L_0}\right)^2$$

So

$$\Delta E = E + \frac{m}{2}\left(\frac{GmM}{L_0}\right)^2$$