# Physics 106b/196b – Problem Set 8 – Due Jan 12, 2007
# Solutions

Andrey Rodionov, Peng Wang, Sunil Golwala

## Problem 1

Let's consider a vector $\vec{r}$ first rotates along $z$- axis by $\theta_1$ and then along $x$-axis by $\theta_2$ where $\theta_1$ and $\theta_2$ are finite angles. So the coordinates of the vector after rotation turn out to be

$$\begin{aligned}\begin{pmatrix}x'\\y'\\z'\end{pmatrix} &= \begin{pmatrix}1 & 0 & 0\\0 & \cos\theta_2 & -\sin\theta_2\\0 & \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\\sin\theta_1 & \cos\theta_1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\&= \begin{pmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\\sin\theta_1\cos\theta_2 & \cos\theta_2\cos\theta_1 & -\sin\theta_2\\\sin\theta_1\sin\theta_2 & \cos\theta_1\sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\end{aligned}$$

On the other hand, if we first rotate along $x$-axis by $\theta_2$ and then along $z$- axis by $\theta_1$, we have

$$\begin{aligned}\begin{pmatrix}x''\\y''\\z''\end{pmatrix} &= \begin{pmatrix}1+\cos\theta_1 & -\sin\theta_1 & 0\\\sin\theta_1 & 1+\cos\theta_1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 1+\cos\theta_2 & -\sin\theta_2\\0 & \sin\theta_2 & 1+\cos\theta_2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\\&= \begin{pmatrix}\cos\theta_1 & -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2\\\sin\theta_1 & \cos\theta_2\cos\theta_1 & -\sin\theta_2\cos\theta_1\\0 & \sin\theta_2 & \cos\theta_2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\end{aligned}$$

If we set $\theta_1 = \theta_2 = \frac{\pi}{2}$, we find

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}1 & -1 & 0\\1 & 1 & -1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$\begin{pmatrix}x''\\y''\\z''\end{pmatrix} = \begin{pmatrix}1 & -1 & 1\\1 & 1 & -1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

where $\begin{pmatrix}1 & -1 & 0\\1 & 1 & -1\\1 & 1 & 1\end{pmatrix} \neq \begin{pmatrix}1 & -1 & 1\\1 & 1 & -1\\0 & 1 & 1\end{pmatrix}$ which means two rotations don't commute. If we have two finite rotations $\mathbf{R}_1 = \exp\left(\vec{\theta}_1 \cdot \vec{\mathbf{M}}\right)$ and $\mathbf{R}_2 = \exp\left(\vec{\theta}_2 \cdot \vec{\mathbf{M}}\right)$, generally, $\left[\exp\left(\vec{\theta}_1 \cdot \vec{\mathbf{M}}\right), \exp\left(\vec{\theta}_2 \cdot \vec{\mathbf{M}}\right)\right]$ won't vanish which means two rotations don't commute. However, if $\vec{\theta}_1$ and $\vec{\theta}_2$ are infinitesimal, we instead have $\mathbf{R}_1 = 1 + \vec{\theta}_1 \cdot \vec{\mathbf{M}}$ and $\mathbf{R}_1 = 1 + \vec{\theta}_2 \cdot \vec{\mathbf{M}}$ and therefore we find $\left[1 + \vec{\theta}_1 \cdot \vec{\mathbf{M}}, 1 + \vec{\theta}_2 \cdot \vec{\mathbf{M}}\right] = 0$

where we we have dropped terms quadratic in the infinitesimal rotation angles, which doesn't happen in the finite rotations case. Just as in our example, when $\vec{\theta}_1$ and $\vec{\theta}_2$ are infinitesimal, one has

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\theta_1 & 0 \\ \theta_1 & 1 & -\theta_2 \\ 0 & \theta_2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} 1 & -\theta_1 & 0 \\ \theta_1 & 1 & -\theta_2 \\ 0 & \theta_2 & \theta_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where we could see the two infinitesimal rotations do commute.

## Problem 2

Matrices $A$ and $B$ are orthogonal matrices.
This means that $AA^T = I$

and $BB^T = I$.

Consider the product $C = AB$.
Computing $C \cdot C^T$, we have:

$$C \cdot C^T = (AB)(AB)^T = A\underbrace{BB^T}A^T = AIA^T =$$
$$= AA^T = I$$

$\Rightarrow CC^T = I$, so the product $C \equiv AB$ of two orthogonal matrices $A$ and $B$ is orthogonal matrix. ▲

Alternative method: we can prove that

$$\sum_i C_{ij} C_{ik} = \delta_{jk} :$$

$$\sum_i C_{ij} C_{ik} = \sum_{ilm} (A_{il} B_{lj})(A_{im} B_{mk}) = \sum_{ilm} (A_{il} A_{im})(B_{lj} B_{mk})$$
$$= \{l = m\} = \sum_{il} B_{lj} \cdot B_{lk} = \delta_{jk} \text{ ▲}$$

## Problem 3

$\vec{c}$ is assumed to be a eigenvector of $\mathbf{R}$ with eigenvalue $\lambda_c$, namely $\mathbf{R}\vec{c} = \lambda_c \vec{c}$ and $\mathbf{R}\vec{c^*} = \lambda_c^* \vec{c^*}$ where we use the fact $\mathbf{R}$ is real. So one easily gets

$$(\mathbf{R}\vec{c})^T \left(\mathbf{R}\vec{c^*}\right) = |\lambda_c|^2 \, \vec{c}^T \, \vec{c^*}$$

On the other hand, we have

$$(\mathbf{R}\vec{c})^T \left(\mathbf{R}\vec{c^*}\right) = \vec{c}^T \mathbf{R}^T \mathbf{R} \vec{c^*} = \vec{c}^T \, \vec{c^*}$$

which we obtain

$$|\lambda_c|^2 = 1$$

Since the vector of rotation axis doesn't change under any rotation in 3D space, the vector must be the eigenvector with eigenvalue 1. Because the determinant of a matrix $\mathbf{M}$ can be written as $\prod_i \lambda_i$ where $\lambda_i$ is a eigenvalue of $\mathbf{M}$. So for an orthogonal matrix $\mathbf{R}$ with determinant=1 is given by $\lambda_1 \lambda_2$ where $\lambda_1, \lambda_2$ are two other eigenvalues of $\mathbf{R}$. Then we get $\lambda_1 \lambda_2 = 1$ which means $\lambda_1, \lambda_2$ can be of the form $\exp(\pm i\alpha)$.

## Problem 4

PROBLEM 4.

$\boxed{1\ \text{part}}$ PROVE the identity:

$$\vec{a}\times[\vec{b}\times\vec{c}] = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Definition of cross-product: $[\vec{a}\times\vec{b}] = \sum_i\sum_j\sum_k \epsilon_{ijk}\,\vec{e}_i A_j B_k$

where $\epsilon_{ijk} = \begin{cases} 1 & \text{for } ijk = 123, \text{ or } 231, \text{ or even permutation} \\ -1 & \text{for odd permutation;} \\ 0 & \text{if at least two indices coincide} \end{cases}$

Using these definitions, consider

$$\vec{W} \equiv [\vec{a}\times[\vec{b}\times\vec{c}]] = \sum_{ijk}\epsilon_{ijk}\,\vec{e}_i\cdot a_j\cdot\{[\vec{b}\times\vec{c}]\}_k =$$

$$= \sum_{ijk}\epsilon_{ijk}\,\vec{e}_i a_j \sum_{lm}\epsilon_{klm} b_l c_m;$$

Sum over "k" involves only two Levi-Civita tensors $\epsilon_{ijk}$ and $\epsilon_{klm}$, so we can use (A.20).

$$\sum_k \epsilon_{ijk}\epsilon_{klm} = \sum_k \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj};$$

(1)

(2)

Consider now $i^{th}$ component of vector $\vec{W}$:

$$W_i = \sum_{jlm}\Big(\sum_k \epsilon_{ijk}\epsilon_{klm}\Big)\cdot a_j b_l c_m =$$

$$= \sum_{jlm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj})\,a_j b_l c_m =$$

$$= \sum_{jlm}\delta_{il}\delta_{jm}\,a_j b_l c_m - \sum_{jlm}\delta_{im}\delta_{lj}\,a_j b_l c_m =$$

$$= \sum_j a_j \underset{(l=i)}{b_i} \underset{(m=j)}{c_j} - \sum_j a_j \underset{(l=j)}{b_j} \underset{(m=i)}{c_i} =$$

$$= \Big(\sum_j a_j c_j\Big) b_i - \Big(\sum_j a_j b_j\Big) c_i = (\vec{a}\cdot\vec{c})b_i - (\vec{a}\cdot\vec{b})c_i;$$

THEREFORE,

$$\boxed{\vec{a}\times[\vec{b}\times\vec{c}] = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})}$$

Next, we want to prove

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{a} \cdot \left[\vec{b} \times (\vec{c} \times \vec{d})\right]$$
$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

The first part is easy because it consists basically of just rearranging the factors and applying cyclicity once:

$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \left(\vec{a} \times \vec{b}\right)_i \left(\vec{c} \times \vec{d}\right)_i = \left(\epsilon_{ijk} a_j b_k\right) \left(\epsilon_{ilm} c_l d_m\right) \\
&= a_j \left(\epsilon_{ijk} b_k \left(\epsilon_{ilm} c_l d_m\right)\right) = a_j \left[\epsilon_{jki} b_k \left(\epsilon_{ilm} c_l d_m\right)\right] \\
&= a_j \left[\epsilon_{jki} b_k \left(\vec{c} \times \vec{d}\right)_i\right] = a_j \left[\vec{b} \times \left(\vec{c} \times \vec{d}\right)\right]_j \\
&= \vec{a} \cdot \left[\vec{b} \times (\vec{c} \times \vec{d})\right]
\end{aligned}$$

where we used the cyclicity property of $\epsilon_{ijk}$ in replacing $\epsilon_{ijk}$ by $\epsilon_{jki}$. The second part can be proven using the same identity as for the proof of the triple product identity:

$$\begin{aligned}
(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \left(\vec{a} \times \vec{b}\right)_i \left(\vec{c} \times \vec{d}\right)_i = \left(\epsilon_{ijk} a_j b_k\right) \left(\epsilon_{ilm} c_l d_m\right) = \left(\epsilon_{ijk} \epsilon_{ilm}\right) \left(a_j b_k c_l d_m\right) \\
&= \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}\right) \left(a_j b_k c_l d_m\right) = a_j c_j b_k d_k - a_j d_j b_k c_k \\
&= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})
\end{aligned}$$

Now, let's consider the case where $\vec{a} = \vec{\nabla}$, $\vec{c} = \vec{\nabla}$, and $\vec{b}$ and $\vec{d}$ are position dependent. The identities tell us

$$\left(\vec{\nabla} \times \vec{b}\right) \cdot \left(\vec{\nabla} \times \vec{d}\right) = \vec{\nabla} \cdot \left[\vec{b} \times \left(\vec{\nabla} \times \vec{d}\right)\right] = \left(\vec{\nabla} \cdot \vec{\nabla}\right) \left(\vec{b} \cdot \vec{d}\right) - \left(\vec{\nabla} \cdot \vec{d}\right) \left(\vec{\nabla} \cdot \vec{b}\right)$$

where, for the second identity, we have made the obvious fix of rewriting $\vec{b} \cdot \vec{c}$ as $\vec{c} \cdot \vec{b}$. According to the first expression, the first $\vec{\nabla}$ should only act on $\vec{b}$ and the second $\vec{\nabla}$ should only act on $\vec{d}$. Our vector identities are entirely algebraic and so should not affect these relationships. But the expressions on the right side do not respect these relationships. In the first expression, the second $\vec{\nabla}$ does indeed only operate on $\vec{d}$, but the first $\vec{\nabla}$ is naturally interpreted as acting on the entire expression $\vec{b} \times \left(\vec{\nabla} \times \vec{d}\right)$, which would result in the first $\vec{\nabla}$ acting on $\vec{d}$. That's nonsense. In the second expression, the first term is clearly nonsense – there is no reason that we should end up with the second-order derivative operator $\vec{\nabla} \cdot \vec{\nabla}$ acting on both $\vec{b}$ and $\vec{d}$ since we started only with separate first-order derivatives. The second term, while it appears sensible, is not because it has the $\vec{\nabla}$ that originally acted on $\vec{b}$ now acting on $\vec{d}$ and the $\vec{\nabla}$ that originally acted on $\vec{d}$ now acting on $\vec{b}$. Clearly, the only way to preserve the relations between the $\vec{\nabla}$ operators and the vectors they should operate on is to use only the index notation and to enforce the operator relationships via parentheses. For example, instead of

$$\left(\vec{\nabla} \times \vec{b}\right) \cdot \left(\vec{\nabla} \times \vec{d}\right) = \vec{\nabla} \cdot \left[\vec{b} \times \left(\vec{\nabla} \times \vec{d}\right)\right]$$

we would have

$$\left(\vec{\nabla} \times \vec{b}\right) \cdot \left(\vec{\nabla} \times \vec{d}\right) = \epsilon_{jki} \epsilon_{ilm} \left(\nabla_j b_k\right) \left(\nabla_l d_m\right)$$

(It was fine to rearrange the parentheses from the expressions given earlier because those parentheses had no impact – they were only there to clarify the situation by grouping factors together.) This expression preserves the operator relationships properly but would be equivalent to the version in vector notation if the $\vec{\nabla}$ vectors were replaced by non-operator vectors.

## Problem 5

**(a)**

$$\begin{aligned}
\underline{\vec{\omega}}'' \cdot \vec{\mathbf{M}} = \dot{\mathbf{R}}\mathbf{R}^T &= \left[\frac{d}{dt}\left(\mathbf{R}_2\mathbf{R}_1\right)\right]\left(\mathbf{R}_2\mathbf{R}_1\right)^T \\
&= \left(\dot{\mathbf{R}}_2\mathbf{R}_1 + \mathbf{R}_2\dot{\mathbf{R}}_1\right)\left(\mathbf{R}_1^T\mathbf{R}_2^T\right) \\
&= \dot{\mathbf{R}}_2\mathbf{R}_2^T + \mathbf{R}_2\dot{\mathbf{R}}_1\mathbf{R}_1^T\mathbf{R}_2^T \\
&= \underline{\vec{\Omega}}_t'' \cdot \vec{\mathbf{M}} + \mathbf{R}_2\underline{\vec{\Omega}}_w' \cdot \vec{\mathbf{M}}\mathbf{R}_2^T \\
&= \underline{\vec{\Omega}}_t'' \cdot \vec{\mathbf{M}} + \mathbf{R}_2\underline{\vec{\Omega}}_w' \cdot \vec{\mathbf{M}}
\end{aligned}$$

The last step requires some further explanation. We can reduce $\mathbf{R}_2\underline{\vec{\Omega}}_w' \cdot \vec{\mathbf{M}}\mathbf{R}_2^T$ in two ways. The first is already discussed in the notes in Section 5.1.4 under **Examples of Tensors** and in Section 5.2.1 under **Acceleration and Fictitious Forces**. In that method, we use the fact that the two rotation matrices in the expression are just transforming the second-rank tensor $\vec{\omega} \cdot \vec{\mathbf{M}}$. Since that second-rank tensor is the contraction of the first-rank tensor $\vec{\omega}$ and the third-rank tensor $\vec{\mathbf{M}}$, one can obtain the transformed version of it by transforming each of these tensors independently and contracting the transformed results. Since $\vec{\mathbf{M}}$ is isotropic (the same before and after rotation transformation), we thus just have to transform $\vec{\omega}$, which is what we did above. The second method is to follow through the details of the index manipulation. Specifically:

$$\left[\mathbf{R}_2\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)\mathbf{R}_2^T\right]_{ij} = \left(\mathbf{R}_2\right)_{ik}\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)_{kl}\left(\mathbf{R}_2^T\right)_{lj}$$

First, let's write out explicitly the $\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)$ term and also rewrite the last factor:

$$\left[\mathbf{R}_2\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)\mathbf{R}_2^T\right]_{ij} = \left(\mathbf{R}_2\right)_{ik}\Omega_{w,m}'\left(\mathbf{M}_m\right)_{kl}\left(\mathbf{R}_2\right)_{jl}$$

Next, let's insert $\mathbf{1} = \mathbf{R}_2^T\mathbf{R}_2$ using index notation via inserting $\delta_{kp}$ and $\delta_{mq}$:

$$\begin{aligned}
\left[\mathbf{R}_2\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)\mathbf{R}_2^T\right]_{ij} &= \left(\mathbf{R}_2\right)_{ik}\delta_{kp}\Omega_{w,m}'\delta_{mq}\left(\mathbf{M}_q\right)_{pl}\left(\mathbf{R}_2\right)_{jl} \\
&= \left(\mathbf{R}_2\right)_{ik}\left(\mathbf{R}_2^T\right)_{ks}\left(\mathbf{R}_2\right)_{sp}\Omega_{w,m}'\left(\mathbf{R}_2^T\right)_{mt}\left(\mathbf{R}_2\right)_{tq}\left(\mathbf{M}_q\right)_{pl}\left(\mathbf{R}_2\right)_{jl} \\
&= \left(\mathbf{R}_2\right)_{ik}\left(\mathbf{R}_2\right)_{sk}\left(\mathbf{R}_2\right)_{sp}\Omega_{w,m}'\left(\mathbf{R}_2\right)_{tm}\left(\mathbf{R}_2\right)_{tq}\left(\mathbf{M}_q\right)_{pl}\left(\mathbf{R}_2\right)_{jl}
\end{aligned}$$

Now, use the orthogonality of $\mathbf{R}_2$ to eliminate the first two occurrences of $\mathbf{R}_2$:

$$\begin{aligned}
\left[\mathbf{R}_2\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)\mathbf{R}_2^T\right]_{ij} &= \delta_{is}\left(\mathbf{R}_2\right)_{tm}\Omega_{w,m}'\left(\mathbf{R}_2\right)_{tq}\left(\mathbf{R}_2\right)_{sp}\left(\mathbf{R}_2\right)_{jl}\left(\mathbf{M}_q\right)_{pl} \\
&= \left[\left(\mathbf{R}_2\right)_{tm}\Omega_{w,m}'\right]\left[\left(\mathbf{R}_2\right)_{tq}\left(\mathbf{R}_2\right)_{ip}\left(\mathbf{R}_2\right)_{jl}\left(\mathbf{M}_q\right)_{pl}\right]
\end{aligned}$$

In the above, we clearly see the rotation transformations acting on the first-rank tensor $\vec{\Omega}$ and the third-rank tensor $\vec{\mathbf{M}}$ separately. Finally, we use the fact that $\vec{\mathbf{M}}$ is isotropic so that it is left unchanged by the three rotation matrices acting on it:

$$\left[\mathbf{R}_2\left(\underline{\vec{\Omega}}_\omega' \cdot \vec{\mathbf{M}}\right)\mathbf{R}_2^T\right]_{ij} = \left[\left(\mathbf{R}_2\right)_{tm}\Omega_{w,m}'\right]\left(\mathbf{M}_t\right)_{ij}$$

or, without index notation,

$$\mathbf{R}_2 \left( \underline{\vec{\Omega}}'_\omega \cdot \vec{\mathbf{M}} \right) \mathbf{R}_2^T = \left( \mathbf{R}_2 \underline{\vec{\Omega}}'_\omega \right) \cdot \vec{\mathbf{M}}$$

Finally, because $\mathbf{M}_x, \mathbf{M}_y$, and $\mathbf{M}_z$ are independent, we have

$$\underline{\vec{\omega}}'' = \underline{\vec{\Omega}}''_t + \mathbf{R}_2 \underline{\vec{\Omega}}'_w$$

**(b)** Assume the train is moving in a circle in $yz$ plane in $F''$ frame and therefore $\vec{\Omega}_t$ is along $x$-axis of $F''$. So the natural coordinate representation $\underline{\vec{\Omega}}''_t$, the coordinates of $\vec{\Omega}_t$ in $F''$ frame, is

$$\underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \\ 0 \\ 0 \end{pmatrix}$$

We also assume that wheels spin in the $xy$ plane in $F'$ frame and $\vec{\Omega}_t$ is along $z$-axis of $F'$ frame. So the natural coordinate representation $\underline{\vec{\Omega}}'_w$, the coordinates of $\vec{\Omega}_w$ in $F'$ frame, is

$$\underline{\vec{\Omega}}'_w = \begin{pmatrix} 0 \\ 0 \\ \Omega_w \end{pmatrix}$$

Frame $F$ is rotating about the $z$-axis of $F'$ with angular velocity $\Omega_w$ relative to $F'$ and so the the rotation matrix $\mathbf{R}_1$ is

$$\mathbf{R}_1 = \begin{pmatrix} \cos \Omega_w t & \sin \Omega_w t & 0 \\ -\sin \Omega_w t & \cos \Omega_w t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Frame $F'$ is rotating about the $x$-axis of $F''$ with angular velocity $\left| \vec{\Omega}_w \right| = \frac{v}{R}$ relative to $F''$ and so the the rotation matrix $\mathbf{R}_2$ is

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{v}{R} t & -\sin \frac{v}{R} t \\ 0 & \sin \frac{v}{R} t & \cos \frac{v}{R} t \end{pmatrix}$$

So we have

$$\mathbf{R}_1^T \underline{\vec{\Omega}}'_w = \begin{pmatrix} 0 \\ 0 \\ \Omega_w \end{pmatrix}$$

$$\mathbf{R}^T \underline{\vec{\Omega}}''_t = \mathbf{R}_1^T \mathbf{R}_2^T \underline{\vec{\Omega}}''_t = \mathbf{R}_1^T \begin{pmatrix} \frac{v}{R} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{v}{R} \cos \Omega_w t \\ \frac{v}{R} \sin \Omega_w t \\ 0 \end{pmatrix}$$

$$\mathbf{R}^T \underline{\vec{\omega}}'' = \mathbf{R}_1^T \underline{\vec{\Omega}}'_w + \mathbf{R}^T \underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \cos \Omega_w t \\ \frac{v}{R} \sin \Omega_w t \\ \Omega_w \end{pmatrix}$$

$$\underline{\vec{\Omega}}'_w = \begin{pmatrix} 0 \\ 0 \\ \Omega_w \end{pmatrix}$$

$$\mathbf{R}_2^T \underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{R}_2^T \underline{\vec{\omega}}'' = \underline{\vec{\Omega}}'_w + \mathbf{R}_2^T \underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \\ 0 \\ \Omega_w \end{pmatrix}$$

$$\mathbf{R}_2 \underline{\vec{\Omega}}'_w = \begin{pmatrix} 0 \\ -\Omega_w \sin \frac{v}{R} t \\ \Omega_w \cos \frac{v}{R} t \end{pmatrix}$$

$$\underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{\vec{\omega}}'' = \mathbf{R}_2 \underline{\vec{\Omega}}'_w + \underline{\vec{\Omega}}''_t = \begin{pmatrix} \frac{v}{R} \\ -\Omega_w \sin \frac{v}{R} t \\ \Omega_w \cos \frac{v}{R} t \end{pmatrix}$$

## Problem 6

The way this problem was written was misleading. The problematic text is

> Show that $\vec{\phi}_1$, $\vec{\phi}_2$, and $\vec{\phi}$ form the sides of a spherical triangle, with the angle opposite to $\vec{\phi}$ determined by the angle between the two axes of rotation $\vec{\phi}_1$ and $\vec{\phi}_2$ (the relationship is not trivial – consider just the simple case of this angle vanishing).

It ought to read

> Show that $\frac{1}{2}\,\vec{\phi}_1$, $\frac{1}{2}\,\vec{\phi}_2$, and $\frac{1}{2}\,\vec{\phi}$ form the sides of a spherical triangle, with the angle opposite to $\frac{1}{2}\,\vec{\phi}$ determined by the angle between the two axes of rotation $\vec{\phi}_1$ and $\vec{\phi}_2$.

I didn't have a chance to do the problem before I assigned it, but I got it out of Goldstein and therefore figured it had been fully vetted. Sad to say, it had not. You might have gotten a hint that some sort of adjustment would be necessary from a similar equation given in the notes relating Euler angles to the angle of rotation about a single axis:

$$
\begin{aligned}
1 + 2\,\cos\Phi &= (1 + \cos\theta)\cos(\phi + \psi) + \cos\theta \\
\cos\left(\frac{\Phi}{2}\right) &= \cos\left(\frac{\phi + \psi}{2}\right)\cos\left(\frac{\theta}{2}\right)
\end{aligned}
$$

We will prove a relation somewhat like the second one.

We will prove the relation using, in fact, a similar technique to the one used to prove the above relation. In that proof, we used the fact that the trace of the rotation written in terms of Euler angles is the same as the trace of the rotation written as a similarity-transformed single-axis rotation, and also the fact that the similarity transformation does not affect the value of the trace. The single-axis rotation matrix gave us the expression on the left involving $\Phi$ for the trace, while the Euler-angle rotation matrix gaves us the corresponding expression on the right for the trace. Our proof will go along similar lines.

Without loss of generality, let us take $\vec{\phi}_1 = \phi_1\,\hat{z}$ and $\vec{\phi}_2 = \phi_2\,(\hat{y}\,\sin\phi_{12} + \hat{z}\,\cos\phi_{12})$ where $\phi_{12}$ is the angle between the two vectors defining the rotation axes. It will always be possible to rotate to a coordinate system in which this is true (first rotate so $\vec{\phi}_1$ is along $\hat{z}$, then do a rotation about $\hat{z}$ so that $\vec{\phi}_2$ is in the $yz$ plane). The same rotation can be applied to the rotation matrix for the single-axis rotation $\vec{\phi}$. This will not change the relationship between the three vectors.

With the above simplification, we may explicitly write $\mathbf{R}_1$ and $\mathbf{R}_2$, the rotation matrices for $\vec{\phi}_1$ and $\vec{\phi}_2$, respectively. $\mathbf{R}_1$ is just a rotation about $\hat{z}$:

$$
\begin{pmatrix}
c_1 & -s_1 & 0 \\
s_1 & c_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(We use the shorthand $c_1 \equiv \cos\phi_1$ and $s_1 \equiv \sin\phi_1$.) We can make a quick check: if $\phi_1 = \frac{\pi}{2}$, then we expect $\hat{x}$ to be rotated into $\hat{y}$ and $\hat{y}$ into $-\hat{x}$. You can check that the above matrix accomplishes this.

$\mathbf{R}_2$ is more complicated. The simplest way to obtain it is to realize that, in a coordinate system in which $\hat{z}$ points along $\vec{\phi}_2$, the rotation will be a simple rotation about $\hat{z}$ like $\mathbf{R}_1$. We can thus obtain $\mathbf{R}_2$ by a similarity transform of a matrix of that form, with the requirement that the rotation matrix $\mathbf{D}$ that does this similarity transform must rotate $\hat{z}$ into $\vec{\phi}_2$ and thus is a rotation about $\hat{x}$

by $-\phi_{12}$:

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{12} & s_{12} \\ 0 & -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{12} & -s_{12} \\ 0 & s_{12} & c_{12} \end{pmatrix}$$

(The same abbreviation scheme holds, with $_2$ corresponding to $\phi_2$ and $_{12}$ corresponding to $\phi_{12}$.) Clearly, the two matrices on either end of the expression are rotations about $\hat{x}$ and, as is necessary for the similarity transform, are transposes of each other. The one on the left is $\mathbf{D}$ and the one on the right $\mathbf{D}^T$. The central matrix is just a rotation about $\hat{z}$ by $\phi_2$. Writing it out, we have

$$\mathbf{R}_2 = \begin{pmatrix} c_2 & c_{12}s_2 & -s_{12}s_2 \\ -c_{12}s_2 & c_{12}^2 c_2 + s_{12}^2 & -c_{12}s_{12}c_2 + c_{12}s_{12} \\ s_{12}s_2 & -c_{12}s_{12}c_2 + c_{12}s_{12} & s_{12}^2 c_2 + c_{12}^2 \end{pmatrix}$$

We can check that this is the correct matrix, up to the sign of the rotation angle $\phi_2$, by just applying it to $\vec{\phi}_2$; that vector should be an eigenvector of $\mathbf{R}_2$ if it is indeed the rotation axis. Specifically,

$$\vec{\phi}_2 = \begin{pmatrix} 0 \\ s_{12} \\ c_{12} \end{pmatrix}$$

By explicitly calculating $\mathbf{R}_2\vec{\phi}_2$, you can see that $\vec{\phi}_2$ is indeed an eigenvector of $\mathbf{R}_2$.

Next, we calculate the product matrix $\mathbf{R} = \mathbf{R}_2\,\mathbf{R}_1$, which is the $\vec{\phi}_1$ rotation followed by the $\vec{\phi}_2$ rotation. Since we will only need the trace of $\mathbf{R}$, we only calculate the diagonal elements:

$$\begin{aligned} \mathbf{R} = \mathbf{R}_2\,\mathbf{R}_1 &= \begin{pmatrix} c_2 & c_{12}s_2 & -s_{12}s_2 \\ -c_{12}s_2 & c_{12}^2 c_2 + s_{12}^2 & -c_{12}s_{12}c_2 + c_{12}s_{12} \\ s_{12}s_2 & -c_{12}s_{12}c_2 + c_{12}s_{12} & s_{12}^2 c_2 + c_{12}^2 \end{pmatrix} \begin{pmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1c_2 + c_{12}s_1s_2 & ? & ? \\ ? & c_{12}s_1s_2 + c_{12}^2 c_1c_2 + s_{12}^2 c_1 & ? \\ ? & ? & s_{12}^2 c_2 + c_{12}^2 \end{pmatrix} \end{aligned}$$

Then, let us take the trace. Some terms cancel, leaving only

$$\begin{aligned} \mathbf{Tr}\,[\mathbf{R}] &= 1 + c_1c_2 + 2\,c_{12}s_1s_2 + c_{12}^2\,(1 + c_1c_2) + s_{12}^2\,(c_1 + c_2) \\ &= 1 + 2\,(c_1c_2 + 2\,c_{12}s_1s_2) + s_{12}^2\,(c_1 + c_2 - 1 - c_1c_2) \end{aligned}$$

where between the two lines we used $c_{12}^2 = 1 - s_{12}^2$. We know from the lecture notes (as indicated above) that the trace of the single-axis rotation version of the matrix is

$$\mathbf{Tr}\left[\mathbf{R}(\vec{\phi})\right] = 1 + 2\,\cos\phi$$

The spherical trigonometry identity we will need is:

$$\cos c = \cos a\,\cos b + \cos C\,\sin a\,\sin b$$

where $a$, $b$, and $c$ are the lengths of the sides of the spherical triangle and $C$ is the angle opposite to side $c$. One can see that the expression we already have is very close if we identify $a = \phi_1$, $b = \phi_2$, $c = \phi$, and $C = \phi_{12}$. But there is the pesky $s_{12}^2$ term. After I banged my head against a wall for a while and completely convinced myself this term is not there erroneously, I was inspired by the half-angle versions of the earlier expression and decided to expand using the half-angle identities:

$$\cos\alpha = 2\,\cos^2\frac{\alpha}{2} - 1 \qquad\qquad \sin\alpha = 2\,\sin\frac{\alpha}{2}\,\cos\frac{\alpha}{2}$$

We will abbreviate the half-angle expressions using $\tilde{c}$ and $\tilde{s}$; that is, $\tilde{c}_a \equiv \cos\frac{\phi_a}{2}$ and $\tilde{s}_a \equiv \sin\frac{\phi_a}{2}$. Let's rewrite our second trace expression first:

$$\mathbf{Tr}\left[\mathbf{R}(\phi)\right] = 1 + 2\left(2\tilde{c}^2 - 1\right) = 4\tilde{c}^2 - 1$$

where $\tilde{c}$ has no subscript because $\phi$ has no subscript. Our first expression requires much more work:

$$\begin{aligned}\mathbf{Tr}\left[\mathbf{R}\right] &= 1 + 2\left(c_1c_2 + 2\,c_{12}s_1s_2\right) + s_{12}^2\left(c_1 + c_2 - 1 - c_1c_2\right)\\ &= 1 + 2\left[\left(2\tilde{c}_1^2 - 1\right)\left(2\tilde{c}_2^2 - 1\right) + 2\,c_{12}\left(2\tilde{s}_1\tilde{c}_1\right)\left(2\tilde{s}_2\tilde{c}_2\right)\right]\\ &\qquad + s_{12}^2\left[\left(2\tilde{c}_1^2 - 1\right) + \left(2\tilde{c}_2^2 - 1\right) - 1 - \left(2\tilde{c}_1^2 - 1\right)\left(2\tilde{c}_2^2 - 1\right)\right]\end{aligned}$$

Multiply everything out and group terms:

$$\mathbf{Tr}\left[\mathbf{R}\right] = 3 - 4s_{12}^2 + \tilde{c}_1^2\tilde{c}_2^2\left(8 - 4s_{12}^2\right) + \left(\tilde{c}_1^2 + \tilde{c}_2^2\right)\left(-4 + 4s_{12}^2\right) + 8c_{12}\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2$$

Make liberal use of $1 - s_{12}^2 = c_{12}^2$:

$$\mathbf{Tr}\left[\mathbf{R}\right] = -1 + 4c_{12}^2 + 4\tilde{c}_1^2\tilde{c}_2^2\left(1 + c_{12}^2\right) - 4c_{12}^2\left(\tilde{c}_1^2 + \tilde{c}_2^2\right) + 8c_{12}\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2$$

Group terms in a suggestive way:

$$\begin{aligned}\mathbf{Tr}\left[\mathbf{R}\right] &= -1 + 4c_{12}^2\left[\tilde{c}_1^2\tilde{c}_2^2 - \left(\tilde{c}_1^2 + \tilde{c}_2^2\right) + 1\right] + 4\tilde{c}_1^2\tilde{c}_2^2 + 8c_{12}\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2\\ &= -1 + 4\left[c_{12}^2\left(1 - \tilde{c}_1^2\right)\left(1 - \tilde{c}_2^2\right) + \tilde{c}_1^2\tilde{c}_2^2 + 2c_{12}\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2\right]\end{aligned}$$

Again use $1 - s_{12}^2 = c_{12}^2$:

$$\begin{aligned}\mathbf{Tr}\left[\mathbf{R}\right] &= -1 + 4\left[c_{12}^2\tilde{s}_1^2\tilde{s}_2^2 + \tilde{c}_1^2\tilde{c}_2^2 + 2c_{12}\tilde{s}_1\tilde{c}_1\tilde{s}_2\tilde{c}_2\right]\\ &= 4\left[\tilde{c}_1\tilde{c}_2 + c_{12}\tilde{s}_1\tilde{s}_2\right]^2 - 1\end{aligned}$$

which is now in similar form to our other trace expression. Equating the two yields

$$\tilde{c}^2 = \left[\tilde{c}_1\tilde{c}_2 + c_{12}\tilde{s}_1\tilde{s}_2\right]^2$$

or, after dropping the square and writing out the expressions explicitly,

$$\cos\frac{\phi}{2} = \cos\frac{\phi_1}{2}\,\cos\frac{\phi_2}{2} + \cos\phi_{12}\,\sin\frac{\phi_1}{2}\,\sin\frac{\phi_2}{2}$$

This is exactly the spherical triangle identity with the identification $a = \frac{1}{2}\,\phi_1$, $b = \frac{1}{2}\,\phi_2$, $c = \frac{1}{2}\,\phi$, and $C = \phi_{12}$. Thus, the desired relation is proven.

## Problem 7

We just need to show how $Z_{i_1\cdots i_{m-1}j_1\cdots j_{n-1}}$ transforms under rotation:

$$\begin{aligned}
Z'_{i_1\cdots i_{m-1}j_1\cdots j_{n-1}} &= X'_{i_1\cdots i_{m-1}s}Y'_{j_1\cdots j_{n-1}s} \\
&= R_{i_1k_1}\cdots R_{i_{m-1}k_{m-1}}R_{sp}X_{k_1\cdots k_{m-1}p}R_{j_1l_1}\cdots R_{j_{n-1}l_{n-1}}R_{sq}Y_{l_1\cdots l_{n-1}q} \\
&= R_{i_1k_1}\cdots R_{i_{m-1}k_{m-1}}\delta_{pq}X_{k_1\cdots k_{m-1}p}R_{j_1l_1}\cdots R_{j_{n-1}l_{n-1}}Y_{l_1\cdots l_{n-1}q} \\
&= R_{i_1k_1}\cdots R_{i_{m-1}k_{m-1}}R_{j_1l_1}\cdots R_{j_{n-1}l_{n-1}}X_{k_1\cdots k_{m-1}p}Y_{l_1\cdots l_{n-1}p} \\
&= R_{i_1k_1}\cdots R_{i_{m-1}k_{m-1}}R_{j_1l_1}\cdots R_{j_{n-1}l_{n-1}}Z_{k_1\cdots k_{m-1}l_1\cdots l_{n-1}}
\end{aligned}$$

which is the appropriate transformation for a tensor of rank $m + n - 2$.