Physics 106b/196b – Problem Set 9 – Due Jan 19, 2007
Solutions

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Note: The TA is double-checking his solution to Problem 6, some of the algebra may not be right.

Problem 1

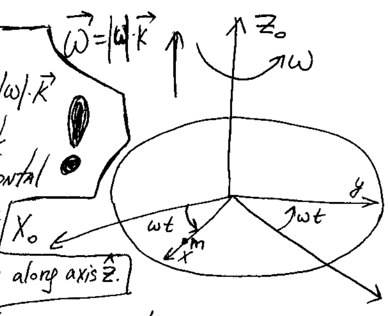
(1)

PROBLEM 1A   A wheel is rotating in the horizontal plane, $\omega = 3.0 \, \frac{rad}{sec}$

$x_0, y_0, z_0 \rightarrow$ axes of inertial laboratory frame

$x, y \rightarrow$ Rotating frame, axes x and y coincide with the spokes.

A bug is crawling along x-axis with constant velocity $v_0 = 0.5 \, \frac{cm}{sec}$



Important Remark: $\vec{\omega} = |\omega| \cdot \vec{k}$!
when wheel/disk rotates in horizontal plane (xy), angular velocity $\vec{\omega}$ is directed vertically along axis $\hat{z}$.

Force acting on the crawling bug is given by (eq. 5.4) of the lecture notes (page 276) or (eq. 7.34) in Hand-Finch.

$$\boxed{\vec{F}_{APPARENT} = \vec{F}_{TRUE} - m[\vec{\omega} \times [\vec{\omega} \times \vec{r}]] - 2m[\vec{\omega} \times \vec{v}_{Body}] - m[\dot{\vec{\omega}} \times \vec{r}]}$$

(2)

As it is explained in lecture notes, it is better to use notation $R^T\vec{\omega}'$ instead of $\vec{\omega}$, but for the simplicity of formulae we will write $\vec{\omega}$ in this solution.

Consider every term in that expression for the force:

- $\omega = \text{const} \Rightarrow \dot{\omega} = 0$, $[\dot{\vec{\omega}} \times \vec{r}] = 0$;
- $\vec{\omega} = \omega \cdot \vec{e}_z$, $\vec{r} = x \cdot \vec{e}_x$;

  $\vec{\omega} \times [\vec{\omega} \times \vec{r}] = -\omega^2 \cdot x \cdot \vec{e}_x$;
- $\vec{\omega} = \omega \vec{e}_z$, $\vec{v}_{body} = v_0 \cdot \vec{e}_x$; $2m[\vec{\omega} \times \vec{v}_{body}] = 2mv_0\omega\vec{e}_y$
- Real, "true" force exerted by the wheel on the bug is $\underbrace{mg\vec{e}_z}_{\substack{\text{Reaction force}\\ \text{which is opposite}\\ \text{to the force "}-mg\vec{e}_z\text{"}\\ \text{(weight of the bug)}}} + \underbrace{\vec{F}_{friction}}_{\substack{\text{if the bug is}\\ \text{not slipping,}\\ |F_{friction}| = \mu mg}}$.

(3)

Therefore, the total force that "feels" the bug is:

$$\vec{F}_{bug} = m\omega^2 x \vec{e}_x - 2mv_0\omega\vec{e}_y + \vec{F}_{friction} + mg\vec{e}_z.$$

Distance X_{MAX}, at which the bug is still not slipping can be determined from the balance of forces in the horizontal plane:

$$\begin{cases} \vec{F}_{friction} = 2mv_0\omega\vec{e}_y - m\omega^2 x\vec{e}_x; \\ |\vec{F}_{friction}| = \mu mg \end{cases}$$

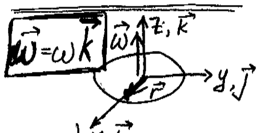
$$\Rightarrow X^2_{MAX} = \left| \frac{4v_0^2\omega^2 - \mu^2 g^2}{\omega^4} \right|;$$

Substituting numerical parameters:

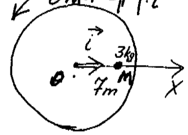
$v_0 = 0{,}5 \frac{cm}{sec} = 0{,}005 \frac{m}{sec}$; $\omega = 3{,}0 \frac{rad}{sec}$; $\mu = 0{,}3$; $g \approx 10 \frac{m}{s^2}$

$$X_{MAX} = \sqrt{\frac{-4\cdot(0{,}005 \times 3{,}0)^2 + (0{,}3 \cdot 10)^2}{81}} \approx 0{,}33\ m = 33\ cm$$

Problem 1b



In horizontal plane:
ω
$\vec{r}=|r|\cdot\vec{i}$



$\vec{i}\,\vec{j}\,\vec{k}$ are unit vectors

Ball with mass $m=3.0\,kg$ at a distance $=7m$ from the center.

We again start from (eq. 5.4) of the lecture notes, or (eq. 7.34) of Hand–Finch.

$$\vec{F}_{APPARENT}=\vec{F}_{TRUE}-m\left[\vec{\omega}\times\left[\vec{\omega}\times\vec{r}\right]\right]-2m\left[\vec{\omega}\times\vec{v}_{Body}\right]-$$
$$-m\left[\dot{\vec{\omega}}\times\vec{r}\right]$$

Consider all the forces in this formula:

→ The ball is held stationary with respect to the carousel $\Rightarrow v_{Body}=0$.

→ $\vec{F}_{TRUE}=\vec{f}-mg\vec{k}$ where $\vec{f}$ is exerted by the girl {we need to find this force}, and "$-mg\vec{k}$" is due to gravity.

→ $\vec{\omega}\perp\vec{r}\Rightarrow\left[\vec{\omega}\times\left[\vec{\omega}\times\vec{r}\right]\right]=-\omega^2\vec{r}$ where $\vec{r}=r\cdot\vec{i}$ is in horizontal plane (Radius-vector from the center of the carousel to the ball).

Explanation:
$\left[\vec{\omega}\times\left[\vec{\omega}\times\vec{r}\right]\right]=\omega^2|r|\cdot\left[\vec{k}\times\left[\vec{k}\times\vec{i}\right]\right]$
$=\omega^2|r|\cdot\left[\vec{k}\times\vec{j}\right]=-\omega^2|r|\cdot\vec{i}$

→ $\left[\dot{\vec{\omega}}\times\vec{r}\right]=\left[\dot{\omega}\vec{k}\times r\vec{i}\right]=\dot{\omega}r\vec{j}$,

because $\dot{\vec{\omega}}=\dot{\omega}\vec{k}$ where $\vec{k}$ is the unit vector along the z-axis that coincides with the rotation axis.

Therefore, $\vec{F}_{APPARENT}=\vec{f}-mg\vec{k}+m\omega^2r\vec{i}-m\dot{\omega}r\vec{j}$

The ball is in equilibrium $\Rightarrow\vec{F}_{APPARENT}=0$.

So, $$\boxed{\vec{f}=-m\omega^2r\vec{i}+m\dot{\omega}r\vec{j}+mg\vec{k}}$$

This formula gives the answer to our problem. Substituting numerical data $\Rightarrow$

$\dot{\omega}=(0.02\times2\pi)\,\frac{Rad}{sec^2}\rightarrow 0.13\,\frac{Rad}{sec^2}$,
$\omega=\dot{\omega}t=6\dot{\omega}\rightarrow 0.78\,\frac{Rad}{sec}$
$m=3.0\,kg$,
$r=7.0\,m$

$$\vec{f}=\left(-13\vec{i}+3\vec{j}+30\vec{k}\right)N$$

($N\equiv$ Newtons)

Problem 2

Let us calculate the coordinates of the center of mass in frame F' in which the origin sits at the center of the base of the cone and z is along the axis of the cone. First we need to calculate the volume of the cone in order to get the density of the cone

$$V = \int_0^h \pi \left[(h - z)\tan\alpha\right]^2 dz = \frac{1}{3}\pi h^3 \tan^2\alpha$$

So the density is $\rho = \frac{M}{\frac{1}{3}\pi h^3 \tan^2\alpha}$. According to the circular symmetry, the x and y coordinates of the center of the mass, x_c and y_c, are zero. And

$$\begin{aligned} z_c &= \int_0^h z\rho\pi \left[(h - z)\tan\alpha\right]^2 dz \\ &= \frac{M}{\frac{1}{3}\pi h^3 \tan^2\alpha}\left(\frac{1}{3}\pi h^4 \tan^2\alpha - \frac{1}{4}\pi h^4 \tan^2\alpha\right) \\ &= \frac{1}{4}h \end{aligned}$$

In the frame F in which the origin sits at the center of mass and z is along the axis of cone, we are going to calculate the moment of inertia tensor.

$$\begin{aligned} I_3 = I_{zz} &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} dz \int_0^{\left(\frac{3}{4}h - z\right)\tan\alpha} rdr \int_0^{2\pi} d\theta\rho\left(x^2 + y^2\right) \\ &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} dz \int_0^{\left(\frac{3}{4}h - z\right)\tan\alpha} rdr \int_0^{2\pi} d\theta\rho r^2 \\ &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} dz \frac{1}{2}\rho\pi\left(\frac{3}{4}h - z\right)^4 \tan^4\alpha \\ &= \frac{\rho\pi}{10}h^5 \tan^4\alpha \\ &= \frac{3}{10}Mh^2 \tan^2\alpha \end{aligned}$$

$$\begin{aligned} I_1 = I_2 = I_{xx} = I_{yy} &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} dz \int_0^{\left(\frac{3}{4}h - z\right)\tan\alpha} rdr \int_0^{2\pi} d\theta\rho\left(z^2 + x^2\right) \\ &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} dz \int_0^{\left(\frac{3}{4}h - z\right)\tan\alpha} rdr \int_0^{2\pi} d\theta\rho\left(z^2 + r^2 \sin^2\theta\right) \\ &= \int_{-\frac{1}{4}h}^{\frac{3}{4}h} \rho dz \left[z^2\pi\left(\frac{3}{4}h - z\right)^2 \tan^2\alpha + \frac{\pi}{4}\left(\frac{3}{4}h - z\right)^4 \tan^4\alpha\right] \\ &= \rho\pi\left[\frac{h^5}{80}\tan^2\alpha + \frac{h^5}{20}\tan^4\alpha\right] \\ &= \frac{3Mh^2}{20}\left[\frac{1}{4} + \tan^2\alpha\right] \end{aligned}$$

where we use

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and $I_{xx} = I_{yy}$ because of the circular symmetry.

Problem 3

(1

Problem 3

Kinetic energy in the non-rotating frame is $T = \frac{1}{2} m \left|\underline{v}'_{space}\right|^2$ (see page 280 of the lecture notes) which is equivalent to

$$T = \frac{1}{2} m \underline{\vec{v}}'^{\,T}_{space} \underline{\vec{v}}'_{space}$$

We know that $\underline{\vec{v}}'_{SPACE}(t) = \underline{\vec{\omega}}'(t) \times \underline{\vec{r}}'(t) + R\,\underline{\vec{V}}_{body}(t)$ (see eq. 5.2 of lecture notes, page 274) where $R \equiv R(t)$ is the rotation matrix and "underline" means coordinate representation of vector.

Therefore, $T = \frac{1}{2} m \left[\underline{\vec{\omega}}' \times \underline{\vec{r}}' + R\,\underline{\vec{V}}_{body}\right]^T \left[\underline{\vec{\omega}}' \times \underline{\vec{r}}' + R\,\underline{\vec{V}}_{body}\right]$

$$= \frac{1}{2} m \left[R^T\left(\underline{\vec{\omega}}' \times \underline{\vec{r}}' + R\,\underline{\vec{V}}_{body}\right)\right]^T \left[R^T\left(\underline{\vec{\omega}}' \times \underline{\vec{r}}' + R\,\underline{\vec{V}}_{body}\right)\right]$$

$$= \frac{1}{2} m \left[\underline{\vec{\omega}} \times \underline{\vec{r}} + \underline{\vec{V}}_{body}\right]^T \left[\underline{\vec{\omega}} \times \underline{\vec{r}} + \underline{\vec{V}}_{body}\right] =$$

$$= \frac{1}{2} m \left[\left|\underline{\vec{V}}_{body}\right|^2 + 2\,\underline{\vec{v}}_{body} \cdot \left[\underline{\vec{\omega}} \times \underline{\vec{r}}\right] + \left[\vec{\omega} \times \vec{r}\right] \cdot \left[\vec{\omega} \times \vec{r}\right]\right] \Longrightarrow$$

(2

Let us switch to the <u>index notation</u>

so we have

$$\boxed{T = \frac{1}{2} m \left\{ \dot{r}_i \dot{r}_i + 2\dot{r}_i \varepsilon_{ijk} \omega_j r_k + \varepsilon_{ijk} \varepsilon_{ilm} \omega_j r_k \omega_l r_m \right\}}$$

→ Lagrangian L coincides with the kinetic energy T.

→ Euler-Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_n} = \frac{\partial L}{\partial r_n}$

→ Firstly, we need to calculate $\frac{\partial L}{\partial \dot{r}_n}$:

$$\frac{\partial L}{\partial \dot{r}_n} = \frac{1}{2} m \frac{\partial}{\partial \dot{r}_n} (\dot{r}_i \dot{r}_i) + m \left(\frac{\partial}{\partial \dot{r}_n} \dot{r}_i \right) \varepsilon_{ijk} \omega_j r_k + 0 =$$

$$= m\dot{r}_n + m\varepsilon_{njk} \omega_j r_k \cdot \delta_{in} = \boxed{m\dot{r}_n + m\varepsilon_{njk} \omega_j r_k} ;$$

As we can see, the canonical momentum $p_n \equiv \frac{\partial L}{\partial \dot{r}_n}$ is not just $m\vec{v}_{body}$, but has more terms:

$$\vec{p} = m\left(\vec{V}_{body} + [\vec{\omega} \times \vec{r}]\right) = m R^T \vec{v}\,'_{space} ;$$

(3)

→ Secondly, we need to calculate $\frac{\partial L}{\partial r_n}$:

$$\left[\frac{\partial L}{\partial r_n} = m\dot{r}_i \varepsilon_{ijk} \omega_j \delta_{kn} + \frac{1}{2} m \varepsilon_{ijk} \varepsilon_{ilm} \left(\omega_j \delta_{kn} \omega_l r_m + \omega_j r_k \omega_l \delta_{mn} \right) \right. ; \Rightarrow$$

now we are ready to write Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_n} = \frac{\partial L}{\partial r_n}$$

$\{ \varepsilon_{ijk} \delta_{kn} = \varepsilon_{ijn} = \varepsilon_{nij} \}$

$$\left[\frac{d}{dt} \left(m\dot{r}_n + m\varepsilon_{njk} \omega_j r_k \right) = m\varepsilon_{nij} \dot{r}_i \omega_j + \frac{1}{2} m \left(\varepsilon_{ijn} \varepsilon_{ilm} \omega_j \omega_l r_m + \varepsilon_{ijk} \varepsilon_{iln} \omega_j r_k \omega_l \right) ; \right.$$

This leads to:

$\{ \varepsilon_{nij} = -\varepsilon_{nji} \}$

$$\left[m\ddot{r}_n = -m\varepsilon_{njk} \dot{\omega}_j r_k - m\varepsilon_{njk} \omega_j \dot{r}_k - m\varepsilon_{nji} \dot{r}_i \omega_j + \frac{1}{2} m \left(-\varepsilon_{nji} \varepsilon_{ilm} \omega_j \omega_l r_m - \varepsilon_{nli} \varepsilon_{ijk} \omega_l \omega_j r_k \right) \right.$$

We used here: $\varepsilon_{ijn} = \varepsilon_{nij} = -\varepsilon_{nji}$ and $\varepsilon_{iln} = -\varepsilon_{nli}$

④

If we look attentively to the last result, we can see that

1) $-m\,\varepsilon_{njk}\,\omega_j\,\dot{r}_k - m\,\varepsilon_{nji}\,\omega_j\,\dot{r}_i = -2m\,[\vec{\omega}\times\dot{\vec{r}}]_n$

2) $\frac{1}{2}\left(-\varepsilon_{nji}\,\varepsilon_{ilm}\,\omega_j\,\omega_l\,r_m - \varepsilon_{nli}\,\varepsilon_{ijk}\,\omega_l\,\omega_j\,r_k\right) =$

$$= \left\{\vec{\omega}\times[\vec{\omega}\times\vec{r}]\right\}_n ;$$

As $\left\{\vec{F}_{\text{Apparent}}\right\}_n \equiv m\ddot{\vec{r}}_n$,

we arrive at

$$\vec{F}_{\text{apparent}}\left(=\sum_n m\ddot{r}_n\,\vec{e}_n\right) = m\ddot{\vec{r}} = -m[\dot{\vec{\omega}}\times\vec{r}] - 2m[\vec{\omega}\times\dot{\vec{r}}] - m[\vec{\omega}\times[\vec{\omega}\times\vec{r}]] \quad \text{OR}$$

$$\boxed{\vec{F}_{\text{Apparent}} = -m[\dot{\vec{\omega}}\times\vec{r}] - 2m[\vec{\omega}\times\vec{V}_{\text{body}}] - m[\vec{\omega}\times[\vec{\omega}\times\vec{r}]]}$$

These are Euler-Lagrange equations.

⑤

Finally, we can calculate Hamiltonian:

$$H = p_n\cdot\dot{r}_n - L = p_n\cdot\left(\frac{p_n}{m} - \varepsilon_{njk}\,\omega_j\,r_k\right) - \underbrace{\frac{1}{2}m\left(\dot{\vec{r}}+[\vec{\omega}\times\vec{r}]\right)^2}_{\equiv T = \frac{p^2}{2m}}$$

we use expression for $\dot{r}_n$ which we derive from our previous result for $p_n = m\dot{r}_n + m\,\varepsilon_{njk}\,\omega_j\,r_k$

$$\Downarrow$$

$$\dot{r}_n = \frac{p_n}{m} - \varepsilon_{njk}\,\omega_j\,r_k$$

$$H = \frac{p^2}{m} - \frac{p^2}{2m} - \left(\vec{p}\cdot[\vec{\omega}\times\vec{r}]\right) =$$

$$= \frac{p^2}{2m} - \left(\vec{\omega}\cdot[\vec{r}\times\vec{p}]\right) = \frac{p^2}{2m} - \left(\vec{\omega}\cdot\vec{\ell}\right) \Rightarrow$$

$$\Rightarrow \boxed{H = H_{\omega=0} - \vec{\omega}\cdot\vec{\ell}_{\text{body}} \quad \text{where } \vec{\ell}_{\text{body}} = \vec{r}\times\vec{p}_{\text{body}}}$$

Problem 4

Just as in the lecture notes, our rotating coordinate system is one fixed to the rotating earth at the location of the diversion channel, with x pointing east, y pointing north, and z normal to the surface. So the angular velocity vector in the rotating system is

$$\omega\left(\widehat{y}\cos\lambda + \widehat{z}\sin\lambda\right)$$

where $\lambda = 60°$. And the velocity of the current in the rotating system is

$$-v\widehat{y}$$

So the Coriolis forces acting on the current are

$$\begin{aligned}&-2m\omega\left(\widehat{y}\cos\lambda + \widehat{z}\sin\lambda\right)\times\left(-v\widehat{y}\right)\\&= -2m\omega v\widehat{x}\sin\lambda\end{aligned}$$

and point to the west. So the water on the west side is highest. The total force acting on the water must be normal to the surface – if it were not, then water would flow parallel to the surface and redistribute itself until this condition is satisfied. So the incline angle of the surface of the water is given by

$$\tan\theta = \frac{F_x}{F_z} = \frac{2m\omega v}{mg} = \frac{2\omega v\sin\lambda}{g} = \frac{2\times 3.4\times\sin 60}{9.8}\frac{2\pi}{24\times 60\times 60} = 4.37\times 10^{-5}$$

The difference between the heights of the two sides is

$$\Delta h = d\tan\theta = 47\times 4.37\times 10^{-5}\ \mathrm{m} = 2\times 10^{-3}\ \mathrm{m}$$

Problem 5

(a) Cone on horizontal surface

At $t = 0$, the cone is lying flat on its side with its apex at the origin and the line of contact coincident with the x' axis. The cone's z axis is its symmetry axis, with $+z$ running from the base to the apex. We define the x and y axes of the body system to be such that the body xz plane coincides with the space $x'z'$ plane at $t = 0$, with xz axes rotated by $\pi/2 + \alpha$ clockwise relative to the $x'z'$ axes. At $t = 0$, the y and y' axes coincide.

The cone rolls without slipping on the plane and returns to its original position in a time τ, i.e. the angular velocity of the center of mass around z'-axis is

$$\vec{\omega}_p = \frac{2\pi}{\tau}\vec{e}_{z'} \equiv \omega_p \vec{e}_{z'} \tag{1}$$

and the cone rolls around its z-axis with angular velocity

$$\vec{\Omega} = \frac{2\pi}{\tau}\frac{1}{\sin\alpha}\vec{e}_z \equiv \Omega\vec{e}_z \tag{2}$$

with $\Omega = \frac{\omega_p}{\sin\alpha}$. Both velocities are indeed positive in sense. Decompose $\vec{\omega}_p$ into the body frame components and compute the *total* angular velocity in the body frame (assuming at initial time the body frame y-axis is on the $x'y'$ plane and the cone is on the x'-axis)

$$\begin{aligned}
\omega_x &= \omega_p \cos\alpha\cos\Omega t = \omega_p\cos\alpha\cos\Omega t \\
\omega_y &= -\omega_p\cos\alpha\sin\Omega t = -\omega_p\cos\alpha\sin\Omega t \\
\omega_z &= \Omega - \omega_p\sin\alpha = \omega_p\left(\frac{1}{\sin\alpha} - \sin\alpha\right)
\end{aligned} \tag{3}$$

The negative sign on ω_y results simply from the way the xy axes rotate about z as the cone rolls: the y axis begins by rotating "down" into the negative z' region. Note also the relative sign of the two pieces contributing to ω_z: this occurs because the z axis points from the base of the cone to the apex and thus makes an angle $> \pi/2$ with the z' axis. The inertia tensor is diagonal in the body frame, so the angular momentum components are trivially

$$\begin{aligned}
L_x &= I_1\omega_x = I_1\omega_p\cos\alpha\cos\Omega t \\
L_y &= I_1\omega_y = -I_1\omega_p\cos\alpha\sin\Omega t \\
L_z &= I_3\omega_z = I_3\omega_p\left(\frac{1}{\sin\alpha} - \sin\alpha\right)
\end{aligned} \tag{4}$$

and the kinetic energy is

$$T = \frac{1}{2}I_1\omega_x^2 + \frac{1}{2}I_1\omega_y^2 + \frac{1}{2}I_3\omega_z^2 = \frac{1}{2}\omega_p^2\left[I_1\cos^2\alpha + I_3\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2\right] \tag{5}$$

Decompose Ω into the space frame components, we have the angular velocity in the space frame

$$\begin{aligned}
\omega_{x'} &= -\Omega\cos\alpha\cos\omega_p t = -\omega_p\cot\alpha\cos\omega_p t \\
\omega_{y'} &= -\Omega\cos\alpha\sin\omega_p t = -\omega_p\cot\alpha\sin\omega_p t \\
\omega_{z'} &= \omega_p - \Omega\sin\alpha = 0
\end{aligned} \tag{6}$$

It is probably counterintuitive that the angular velocity along the z' axis vanishes! One can understand this by realizing that the total motion is just rotation about the instantaneous line of contact between the cone and the plane, which always is in the $x'y'$ plane. Thus, $\omega_{z'}$ vanishes. What about the space-frame angular momentum? At the initial time, the total angular momentum in the space frame is in the xz plane and $x'z'$ plane. The total angular momentum in the space frame is

$$\begin{aligned} \vec{L} &= I_1\omega_x\vec{e}_x + I_3\omega_z\vec{e}_z = I_1\omega_p\cos\alpha\vec{e}_x + I_3\omega_p\left(\frac{1}{\sin\alpha} - \sin\alpha\right)\vec{e}_z \\ &= I_1\omega_p\cos\alpha\left(-\sin\alpha\vec{e}_{x'} + \cos\alpha\vec{e}_{z'}\right) + I_3\omega_p\left(\frac{1}{\sin\alpha} - \sin\alpha\right)\left(-\cos\alpha\vec{e}_{x'} - \sin\alpha\vec{e}_{z'}\right) \end{aligned} \tag{7}$$

The total angular momentum is precessing around the z'-axis, so it is given in time as

$$\begin{aligned} \vec{L} &= I_1\omega_p\cos\alpha\left(-\sin\alpha\cos\omega_p t\,\vec{e}_{x'} - \sin\alpha\sin\omega_p t\,\vec{e}_{y'} + \cos\alpha\vec{e}_{z'}\right) \\ &\quad + I_3\omega_p\left(\frac{1}{\sin\alpha} - \sin\alpha\right)\left(-\cos\alpha\cos\omega_p t\,\vec{e}_{x'} - \cos\alpha\sin\omega_p t\,\vec{e}_{y'} - \sin\alpha\vec{e}_{z'}\right) \end{aligned} \tag{8}$$

The total angular momentum is not constant for general α, i.e. external torque is necessary to enforce this motion.

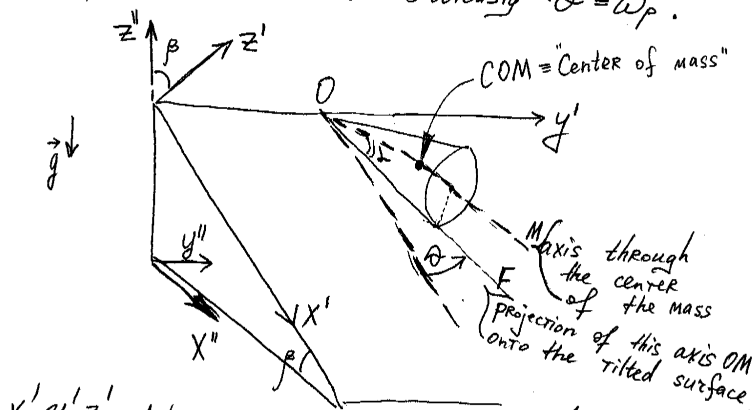
The kinetic energy is in general $T = \frac{1}{2}\left(\vec{\omega}\right)^T \mathcal{I}\vec{\omega}$; since we have $\vec{\omega}$ and $\vec{L}$, though, it will be easier to make use of the equivalent form $T = \frac{1}{2}\left(\vec{\omega}\right)^T \vec{L}$, which gives

$$T = \frac{1}{2}\omega_p^2\left[I_1\cos^2\alpha + I_3\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2\right] \tag{9}$$

which equals the kinetic energy in the body frame (Eq. (5)).

Note that there was no need to explicitly add in center-of-mass motion because in (a) we calculated the inertia tensor relative to the apex of the cone, not the center of mass. Had we calculated the inertia tensor relative to the center of mass, we would have had to include the additional center of mass motion. Calculating the inertia tensor relative to a nonstandard point can thus simplify some problems.

Problem 5b Cone on the tilted plane.
The tilted plane forms angle β with the horizontal surface.
The apex of the cone is fixed, so the only generalized coordinate in this part of the problem is θ. Obviously $\dot{\theta} = \omega_p$.



$x'y'z'$ determine the frame of tilted surface $x'y'$ and axis z' is orthogonal to $x'y'$, axis z' is not vertical; there is angle β between axis z' and vertical axis z''.
$x''y''z''$ determine the system (frame) where gravity $\vec{g}$ acts along vertical axis z''.
Angle θ is in the plane $x'y'$; OF is the projection of axis OM onto the plane $x'y'$. Axis OM is through the COM.

(-2-)

We know from problem 2 of this PS9 that the center of mass of the cone is on its axis at a distance $\frac{3}{4}h$ along the axis of the cone.

Coordinates of the center of mass are:

$$\vec{r}_{CM}(t) = \frac{3}{4}h\cdot\cos\alpha\left(\hat{x}'\cdot\cos\theta(t) + \hat{y}'\cdot\sin\theta(t)\right) + \frac{3}{4}h\cdot\sin\alpha\cdot\hat{z}'\,;$$

We need to transform this result to the frame $(x''y''z'')$ in which gravity is acting along z''-axis.

$$\begin{pmatrix}\hat{x}'\\ \hat{z}'\end{pmatrix} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\hat{x}''\\ \hat{z}''\end{pmatrix};$$

axis $\hat{y}'$ is parallel to axis $\hat{y}''$.

$$\Rightarrow \vec{r}_{CM}(t) = \frac{3}{4}h\Big[\cos\alpha\left\{(\hat{x}''\cos\beta - \hat{z}''\sin\beta)\cdot\cos\theta(t)\right\} + \hat{y}''\cdot\sin\theta(t)\cdot\cos\alpha + (\hat{x}''\sin\beta + \hat{z}''\cos\beta)\cdot\sin\alpha\Big];$$

We need z''-coordinate of the center of mass, because potential energy $V(\theta) = Mg\cdot z''_{CM}(\theta)$

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$$Z_{cm}(\theta) = -\frac{3}{4}h\cdot\cos\alpha\cdot\sin\beta\cdot\cos\theta(t) + \frac{3}{4}h\cdot\sin\alpha\cdot\cos\beta\,,$$

but time-dependence is contained in the first addend only, therefore

$$Z_{cm}(\theta) = -\frac{3}{4}h\cdot\cos\alpha\cdot\sin\beta\cdot\cos\theta(t) + Z_0$$

and potential energy takes the form

$$\boxed{V(\theta) = -\frac{3}{4}Mgh\cdot\cos\alpha\cdot\sin\beta\cdot\cos\theta(t)}\,,$$

Kinetic energy was found in part (A):

$$T(\dot\theta) = \frac{1}{2}\omega_p^2\left[\tilde I_1\cdot\cos^2\alpha + I_3\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2\right],$$

or, introducing notation $\boxed{K \equiv \tilde I_1\cdot\cos^2\alpha + I_3\cdot\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2}$ (K = constant combination of moments of inertia)

and using the fact that $\omega_p \equiv \dot\theta$

we have $\boxed{T(\dot\theta) = \frac{K}{2}\dot\theta^2}$

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Therefore, Lagrange's function is:

$$\boxed{L = T - V = \frac{K}{2}\dot\theta^2 + \frac{3}{4}Mgh\cos\alpha\cdot\sin\beta\cdot\cos\theta(t)};$$

Euler-Lagrange equation of motion is:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0$$

$$K\ddot\theta + \frac{3}{4}Mgh\cdot\cos\alpha\cdot\sin\beta\cdot\sin\theta(t) = 0\,;$$

Small-angle approximation: $\sin\theta(t) \approx \theta(t)$.

$$\ddot\theta + \frac{3}{4}\frac{Mgh\cdot\cos\alpha\cdot\sin\beta}{\tilde I_1\cdot\cos^2\alpha + I_3\cdot\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2}\,\theta = 0\,,$$

and the equation of harmonic oscillations is $\ddot\theta + \omega^2\theta = 0$;

Therefore, the frequency of small oscillations is given by

$$\boxed{\omega = \left[\frac{3}{4}\frac{Mgh\cos\alpha\cdot\sin\beta}{\tilde I_1\cdot\cos^2\alpha + I_3\left(\frac{1}{\sin\alpha} - \sin\alpha\right)^2}\right]^{\frac{1}{2}}}$$

Problem 6

(a) The system contains be a collection of particles with positions $\{\vec{r}_a\}$ and masses $\{m_a\}$. In the lecture notes, one has

$$\underline{\vec{v}_a}'_{space} = \underline{\vec{\Omega}}' \times \underline{\vec{r}_a}' + \mathbf{R}(t)\underline{\vec{v}_a}_{body}$$

. So the kinetic energy is

$$\begin{aligned}
T' &= \sum_a \frac{1}{2} m_a \left(\underline{\vec{v}_a}'_{space}\right)^2 = \sum_a \frac{1}{2} m_a \left(\underline{\vec{\Omega}}' \times \underline{\vec{r}_a}' + \mathbf{R}(t)\underline{\vec{v}_a}_{body}\right)^2 \\
&= \sum_a \frac{1}{2} m_a \left(\underline{\vec{v}_a}_{body}\right)^2 + \sum_a \frac{1}{2} m_a \left(\underline{\vec{\Omega}}' \times \underline{\vec{r}_a}'\right)^2 + \sum_a m_a \left(\underline{\vec{v}_a}_{body} \cdot \left(\mathbf{R}^T(t)\underline{\vec{\Omega}}' \times \underline{\vec{r}_a}'\right)\right) \\
&= T + \sum_a \frac{1}{2} m_a \left(\underline{\vec{\Omega}} \times \underline{\vec{r}_a}\right)^2 + \sum_a m_a \left(\underline{\vec{v}_a}_{body} \cdot \left(\underline{\vec{\Omega}} \times \underline{\vec{r}_a}\right)\right) \\
&= T + \sum_a \frac{1}{2} m_a \left(\underline{\Omega}^2 \underline{r_a}^2 - \left(\underline{\vec{\Omega}} \cdot \underline{\vec{r}_a}\right)\right)^2 + \sum_a m_a \left(\underline{\vec{\Omega}} \cdot \left(\underline{\vec{r}_a} \times \underline{\vec{v}_a}_{body}\right)\right) \\
&= T + \frac{1}{2}\underline{\vec{\Omega}} \cdot \underline{\mathcal{I}} \cdot \underline{\vec{\Omega}} + \underline{\vec{\Omega}} \cdot \underline{\vec{L}}
\end{aligned}$$

where $T = \sum_a \frac{1}{2} m_a \left(\underline{\vec{v}_a}_{body}\right)^2$, $\mathbf{R}^T(t)\underline{\vec{\Omega}}' \times \underline{\vec{r}_a}' = \underline{\vec{\Omega}} \times \underline{\vec{r}_a}$ and $\underline{\vec{v}_a}_{body} \cdot \left(\underline{\vec{\Omega}} \times \underline{\vec{r}_a}\right) = \underline{\vec{\Omega}} \cdot \left(\underline{\vec{r}_a} \times \underline{\vec{v}_a}_{body}\right)$. The Lagrangian is

$$L = T' - V = T + \frac{1}{2}\underline{\vec{\Omega}} \cdot \underline{\mathcal{I}} \cdot \underline{\vec{\Omega}} + \underline{\vec{\Omega}} \cdot \underline{\vec{L}} - V$$

The two additional terms are essentially the contributions to the kinetic energy that come about because the unprimed frame is moving. To get the kinetic energy relative to the unprimed frame from the angular velocity relative to the primed frame, one needs to add the angular velocity at which the primed frame moves relative to the unprimed frame. Since the kinetic energy is quadratic in the angular velocity, the additional angular velocity yields one term that is quadratic in this angular velocity and one term that is the cross-term between the angular velocity relative to the primed frame and the angular velocity of the primed frame.

(b) In the top frame F'', the moment of inertia tensor is

$$\underline{\mathcal{I}}'' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_2 \end{pmatrix}$$

The rotation matrix R from F'' to F is given by

$$R = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where θ is the angle between z'' and z and ψ is the angle by whch the top rotates around the symmetric axis. So

$$\begin{aligned}
\underline{\mathcal{I}} &= R\underline{\mathcal{I}}'' R^T \\
&= \begin{pmatrix} I_1\cos^2\theta + I_2\sin^2\theta & 0 & (I_1 - I_2)\cos\theta\sin\theta \\ 0 & I_1 & 0 \\ (I_1 - I_2)\cos\theta\sin\theta & 0 & I_2\cos^2\theta + I_1\sin^2\theta \end{pmatrix}
\end{aligned}$$

In order to use the angular velocity for the Euler angles, $x \leftrightarrow y, \psi \leftrightarrow -\psi$, and $\phi = 0$. So

$$\underline{\overrightarrow{\omega}} = \begin{pmatrix} \dot{\psi}\sin\theta \\ \dot{\theta} \\ \dot{\psi}\cos\theta \end{pmatrix}$$

$$\begin{aligned} T &= \frac{1}{2}\underline{\overrightarrow{\omega}}^T \underline{\mathcal{I}}\underline{\overrightarrow{\omega}} \\ &= \frac{1}{4}\left[2\dot{\theta}^2 I_1 + \dot{\psi}^2\left((I_1+I_2)+(I_2-I_1)\cos^4\theta\right)\right] \end{aligned}$$

We assume $\underline{\overrightarrow{\Omega}} = \Omega_y\widehat{y} + \Omega_z\widehat{z}$ and then

$$\frac{1}{2}\underline{\overrightarrow{\Omega}}\cdot\underline{\mathcal{I}}\cdot\underline{\overrightarrow{\Omega}} = \frac{1}{2}\left[I_1\Omega_y^2 + \left(I_2\cos^2\theta + I_1\sin^2\theta\right)\Omega_z^2\right]$$

$$\begin{aligned} \underline{\overrightarrow{\Omega}}\cdot\underline{\overrightarrow{L}} &= \underline{\overrightarrow{\Omega}}^T\underline{\mathcal{I}}\underline{\overrightarrow{\omega}} \\ &= I_1\dot{\theta}\Omega_y + \dot{\psi}\Omega_z\left(I_2\cos^2\theta + (2I_1 - I_2)\cos\theta\sin^2\theta\right) \end{aligned}$$

So The Lagrangian is

$$\begin{aligned} L = &\frac{1}{4}\left[2\dot{\theta}^2 I_1 + \dot{\psi}^2\left((I_1+I_2)+(I_2-I_1)\cos^4\theta\right)\right] + \frac{1}{2}\left[I_1\Omega_y^2 + \left(\cos^2\theta I_2 + \sin^2\theta I_1\right)\Omega_z^2\right] + I_1\dot{\theta}\Omega_y \\ &+ \dot{\psi}\Omega_z\left(I_2\cos^2\theta + (2I_1-I_2)\cos\theta\sin^2\theta\right) \\ &\sim \frac{1}{2}\left[\dot{\theta}^2 I_1 + \dot{\psi}^2\left(I_2 + 4(I_1-I_2)\theta^2\right)\right] + \frac{1}{2}\left[I_1\Omega_y^2 + \left(I_2 + \theta^2\left(I_1 - \frac{I_2}{2}\right)\right)\Omega_z^2\right] + I_1\dot{\theta}\Omega_y \\ &+ \dot{\psi}\Omega_z\left(I_2 + \left(2I_1 - \frac{3}{2}I_2\right)\theta^2\right) \end{aligned}$$

Since $\frac{\partial L}{\partial\psi} = 0$, p_ψ =constant and we have

$$\begin{aligned} p_\psi &= 2\left(I_2 + 4(I_1-I_2)\theta^2\right) + \Omega_z\left(I_2 + \left(2I_1 - \frac{3}{2}I_2\right)\theta^2\right) = \Omega_z C \\ \dot{\psi} &= -\frac{\Omega_z\left(I_2 + \left(2I_1 - \frac{3}{2}I_2\right)\theta^2\right) - \Omega_z C}{2\left(I_2 + 4(I_1-I_2)\theta^2\right)} \\ &\sim -\frac{\Omega_z}{2}\left[1 - \frac{C}{I_2} + \left(\frac{5}{2} - 2\frac{I_1}{I_2} - \frac{4C}{I_2} + \frac{4CI_1}{I_2^2}\right)\theta^2\right] \end{aligned}$$

where C is a constant determined by initial conditions. The EOM for θ is

$$\begin{aligned} \frac{d}{dt}\left(2I_1\dot{\theta} + I_1\Omega_y\right) - \dot{\psi}^2\left(8(I_1-I_2)\theta\right) - \theta\left(I_1 - \frac{I_2}{2}\right)\Omega_z^2 - 2\dot{\psi}\Omega_z\left(2I_1 - \frac{3}{2}I_2\right)\theta &= 0 \\ 2I_1\ddot{\theta} - 2\Omega_z^2\left[1 - \frac{C}{I_2}\right]^2(I_1-I_2)\theta - \theta\left(I_1 - \frac{I_2}{2}\right)\Omega_z^2 + \Omega_z^2\left[1 - \frac{C}{I_2}\right]\left(2I_1 - \frac{3}{2}I_2\right)\theta &= 0 \\ \ddot{\theta} + \Omega_z^2\theta\left[\left(1 - \frac{C}{I_2}\right)^2\left(\frac{I_2}{I_1} - 1\right) + \left(\frac{I_2}{2I_1} - 1\right) + \left(1 - \frac{C}{I_2}\right)\left(2 - \frac{3}{2}\frac{I_2}{I_1}\right)\right] &= 0 \end{aligned}$$

So the frequency is

$$\omega^2 = \Omega_z^2\left[\left(1 - \frac{C}{I_2}\right)^2\left(\frac{I_2}{I_1} - 1\right) + \left(\frac{I_2}{2I_1} - 1\right) + \left(1 - \frac{C}{I_2}\right)\left(2 - \frac{3}{2}\frac{I_2}{I_1}\right)\right]$$

Problem 7

(a) Consider a small mass element in the sphere at position $\vec{r}$, its velocity is $\vec{v} = \vec{\omega} \times \vec{r}$, and the torque is

$$\begin{aligned} d\vec{N} &= \vec{r} \times \vec{F} = \frac{q(r)}{c} \vec{r} \times \left(\vec{v} \times \vec{B}\right) = \frac{q(r)}{c} \vec{r} \times \left[(\vec{\omega} \times \vec{r}) \times \vec{B}\right] \\ &= \frac{q(r)}{c} \vec{r} \times \left[\left(\vec{\omega} \cdot \vec{B}\right) \vec{r} - \left(\vec{r} \cdot \vec{B}\right) \vec{\omega}\right] = \frac{q(r)}{c} (\vec{\omega} \times \vec{r}) \left(\vec{r} \cdot \vec{B}\right) \end{aligned} \tag{10}$$

where $q(r)$ is the charge density. Integrate over the body

$$\vec{N} = \int \frac{q(r)}{c} (\vec{\omega} \times \vec{r}) \left(\vec{r} \cdot \vec{B}\right) d\tau = \vec{\omega} \times \left(\int \frac{q(r)}{c} \vec{r}\vec{r}^T d\tau\right) \cdot \vec{B} = \frac{I_e}{c} \vec{\omega} \times \vec{B}$$

where we used the fact that the charge distribution is spherically symmetric, so the representation of the tensor $\int \frac{q(r)}{c} \vec{r}\vec{r}^T d\tau$ in a Cartesian frame with origin at the center of the sphere is proportional to the identity matrix, and the coefficient is defined as

$$I_e \equiv \int q(r) x^2 d\tau = \int q(r) y^2 d\tau = \int q(r) z^2 d\tau \tag{11}$$

For the same reason, the representation of the momentum of inertia tensor is also proportional to identity matrix, with coefficient

$$\begin{aligned} I &= \int \rho(r)(y^2 + z^2) d\tau = \int \rho(r)(z^2 + x^2) d\tau = \int \rho(r)(x^2 + y^2) d\tau \\ &= 2\int \rho(r) x^2 d\tau = 2\int \rho(r) y^2 d\tau = 2\int \rho(r) z^2 d\tau \end{aligned} \tag{12}$$

and we have

$$\vec{N} = \frac{1}{c} \frac{I_e}{I} \vec{L} \times \vec{B} = \frac{qg}{2mc} \vec{L} \times \vec{B} \tag{13}$$

where the gyromagnetic ratio is given by

$$g = \frac{2m}{q} \frac{I_e}{I} = \frac{m}{q} \frac{\int q(r) x^2 d\tau}{\int \rho(r) x^2 d\tau} \tag{14}$$

(b) If the mass density is everywhere proportional to the charge density, that is to say $q(r) = \dfrac{q}{m} \rho(r)$, we have $g = 1$ according to Eq. (14).

(c) The equation of motion of the angular momentum is already derived in (a), i.e.

$$\frac{d\vec{L}}{dt} = \vec{N} = \frac{qg}{2mc} \vec{L} \times \vec{B} \tag{15}$$

Since $\vec{L}$ is a vector, in a frame rotating with constant angular momentum $\vec{\omega}_0$, the angular momentum evolution is

$$\left.\frac{d\vec{L}}{dt}\right|_{\text{rot}} = \frac{d\vec{L}}{dt} + \vec{\omega}_0 \times \vec{L} = \left(\vec{\omega}_0 - \frac{qg}{2mc} \vec{B}\right) \times \vec{L} \tag{16}$$

so the effect of magnetic torque is eliminated seen in a frame rotating with angular velocity $\vec{\omega}_0 = \dfrac{qg}{2mc} \vec{B}$.

(d) The equation of motion of the angular momentum tells us that $\vec{L}$ itself rotates with angular velocity $\vec{\omega}_L = \frac{qg}{2mc}\vec{B}$. That is, the sphere picks up an additional angular velocity $\vec{\omega}_L$ along $\vec{B}$, causing the 3-axis (which is defined to be the axis along which $\vec{L}$ was originally pointing, though, of course, one can pick any direction for the 3-axis of a sphere) to precess at angular velocity $\vec{\omega}_L$ about $\vec{B}$. Since the object is spherically symmetric, the additional angular momentum $I\vec{\omega}_L$ is also along $\vec{B}$, so $I\vec{\omega}_L \times \vec{B} = 0$ and thus the additional torque vanishes.

(e) Let us consider the evolution of $\vec{L} \cdot \vec{v}$, where $\vec{v}$ is the linear velocity

$$\begin{aligned} \frac{d}{dt}\left(\vec{L} \cdot \vec{v}\right) &= \frac{d\vec{L}}{dt} \cdot \vec{v} + \vec{L} \cdot \frac{d\vec{v}}{dt} \\ &= \frac{q}{mc}\left[\left(\vec{L} \times \vec{B}\right) \cdot \vec{v} + \vec{L} \cdot \left(\vec{v} \times \vec{B}\right)\right] = 0 \end{aligned} \tag{17}$$

where the last expression vanishes because the triple vector product is cyclic. *i.e.*, $\vec{L} \cdot \vec{v}$ is a constant. We have shown earlier that the spin angular momentum of an electron is constant in magnitude under the influence of a magnetic field that is constant over the electron. You also know that, because the magnetic force is perpendicular to $\vec{v}$, it does not change the magnitude of $\vec{v}$ either (ignoring any EM wave radiation during the motion of the electron). Therefore, the constancy of $\vec{L} \cdot \vec{v}$ ensures that the angle between the two is constant, and so $\vec{L}$ is always aligned with $\vec{v}$ if it is so initially.

Another way of saying the same thing is that we know the following (for $g = 2$):

$$\frac{d\vec{L}}{dt} = \frac{q}{mc}\vec{L} \times \vec{B} \qquad \frac{d\vec{v}}{dt} = \frac{q}{mc}\vec{v} \times \vec{B} \tag{18}$$

The first equation was part (a) of this problem, the second equation is the Lorentz force. As noted above, these kinds of cross-product rates of change imply that $\vec{L}$ and $\vec{v}$ do not change in length, but simply rotate about $\vec{B}$. The angular rotation rates are

$$\vec{\omega}_L = \frac{1}{|\vec{L}|}\frac{d\vec{L}}{dt} = \frac{q}{mc}\frac{\vec{L}}{|\vec{L}|} \times \vec{B} \qquad \vec{\omega}_v = \frac{1}{|\vec{v}|}\frac{d\vec{v}}{dt} = \frac{q}{mc}\frac{\vec{v}}{|\vec{v}|} \times \vec{B} \tag{19}$$

which are equal if the two vectors point along the same direction. That is, the two vector precess at the same rate, ensuring that they stay aligned if the start out aligned.