TEP4145: Oppsummering 4.5 + 9.5. 2007

KAP 1: Fundamentale prinsipper

- rep. fra TFY4145 / FY1001:

$$\vec{F} = \dot{\vec{p}}, \quad \vec{N} = \dot{\vec{L}}, \quad \vec{F} = -\nabla V \quad (\text{konservativt system})$$

- system med flere partikler:

  ytre og indre krefter: $\vec{F}_i^{(e)}$, $\vec{F}_{ij}$

$$\vec{F}^{(e)} = M\ddot{\vec{R}} = \dot{\vec{P}} \quad (\vec{R} = \text{CM},\ \vec{P} = \text{total impuls})$$

$$\vec{N}^{(e)} = \dot{\vec{L}} \quad (\vec{L} = \text{total dreieimpuls})$$

- føringer:

  holonome: $f(\vec{r}_1, \vec{r}_2, \dots, t) = 0$

  rheonome: tidsavhengige

  skleronome: tidsuavhengige

- generaliserte koord.

  N partikler, k holonome føringer ⇒ 3N−k frihetsgrader,

  ⇒ 3N−k uavh. koord. $q_1 \cdots q_{3N-k}$

- D'Alemberts prinsipp:

$$\sum_i \Big(\underset{\substack{\uparrow \\ \text{ytre kraft}}}{\vec{F}_i^{(a)}} \underbrace{- \dot{\vec{p}}_i}_{\substack{\uparrow \\ \text{effektiv motkraft}}} \Big) \cdot \underset{\substack{\uparrow \\ \text{virtuell forskyvning}}}{\delta \vec{r}_i} = 0$$

- generaliserte krefter:

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- Lagranges ligninger (holonomt ~~[illegible]~~ system):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

Kons. system $\Rightarrow Q_j = -\partial V/\partial q_j$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad ; \quad L = T - V = L(q, \dot{q}, t)$$

L ikke entydig: $L' = L + \frac{dF(q,t)}{dt} \Rightarrow$ samme beveg. lign.

- generaliserte potensialer:

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} \quad ; \quad L = T - U \; ; \quad U = U(q, \dot{q})$$

Eks: E.m. potensial $U = q\phi - q\vec{A}\cdot\vec{v} = U(x_i, v_i)$

ladning!

- friksjon:

$$F_{fx} = -k_x v_x = -\frac{\partial}{\partial v_x}\left(\frac{1}{2}k_x v_x^2\right)$$

$$\vec{F}_f = -\nabla_v \mathcal{F} \; ; \quad \mathcal{F} = \frac{1}{2}\sum_i \left(k_x v_{ix}^2 + k_y v_{iy}^2 + k_z v_{iz}^2\right)$$

$$Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0$$

KAP 2: Variasjonsprinsipp

- Hamiltons prinsipp: $\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0$

$$\Downarrow$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \; ; \quad L = T - V \text{ evt } T - U, \text{ med}$$

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i}$$

- variasjonsregning generelt:

$$\delta I = \int_{x_1}^{x_2} f(y, y', x)\, dx = 0 \;\Rightarrow \text{Eulers ligninger } \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

- ikke-holonome systemer: ikke alle δq_k uavhengige

$\Rightarrow$ tyr til "Lagranges metode med ubestemte koeffisienter"

- bevaringslover og symmetriegenskaper:

  L uavh. av q_i $\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \Rightarrow \dot{p}_i = 0$ (q_i syklisk)

  Kanonisk impuls: $p_i \equiv \partial L / \partial \dot{q}_i$

  Invarians mhp translasjon $\Rightarrow \vec{P}$ konst.

  —— " —— rotasjon $\Rightarrow \vec{L}$ konst. omkring rot.aksen

  Energibevarelse: $\partial L / \partial t = 0 \Rightarrow dH/dt = 0$;

$$H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = T + V = E$$

KAP 8: Hamiltons ligninger

- $\underbrace{H = p_i \dot{q}_i - L}_{}$; $H = H(q, p, t)$ mens $L = L(q, \dot{q}, t)$

  Legendretransformasjon $\Rightarrow$ variabelskifte, fra $(q, \dot{q}, t)$ til (q, p, t)

- Hamiltons ligninger: $\dot{q}_i = \frac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\quad \left(-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}\right)$

- Sentralfelt: $H(q,p) = \frac{1}{2m}\left(p_r^2 + p_\theta^2/r^2 + p_\varphi^2/r^2\sin^2\theta\right) + V(r)$
- E.m. felt: $H(x_i, p_i, t) = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$

KAP 3: Sentralfelt; 2-Legemeproblemet

- 2 legemer + sentralfelt $V(r)$ $\longrightarrow$ ekvivalent 1-legemeproblem:

  $$L = T - V = \frac{1}{2}\mu \dot{\vec{r}}^2 - V(r)$$

  red. masse (μ) — relativkoord. ($\vec{r}$)

  $\vec{L}$ bevart $\Rightarrow$ plan bevegelse: $\vec{r} \to (r, \theta)$

  $r(t)$ gitt ved $\quad t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E - V - \ell^2/2mr^2)}}$

  $r(\theta)$ —"— $\quad \theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - \frac{1}{r^2}}}$

- Kepler-problemet: $V = -k/r \Rightarrow r = \frac{p}{1 + \varepsilon\cos\theta}$ (kjeglesnitt)

- Spredning i sentralfelt: $\sigma(\theta) \sim 1/\sin^4\theta/2$ (Rutherford)

- Spredningstverrsnitt:

  $\sigma(\Omega)\,d\Omega$ = antall partikler spredt inn i $d\Omega$ pr tidsenhet / innfallende intensitet

  $\sigma = \int \sigma(\Omega)\,d\Omega$ = totalt spr.tr.snitt

KAP 4: Stive legemers kinematikk

- stivt legeme $\Rightarrow$ 6 frihetsgrader: $\vec{R}$ (CM) + <u>orientering</u> i rommet (f.eks. Eulervinklene φ, θ, ψ)

- ortogonale transformasjoner: $\vec{r}\,' = \mathbb{A}\vec{r}$, evt $\mathbb{x}' = \mathbb{A}\mathbb{x}$

$$\mathbb{A}^{-1} = \tilde{\mathbb{A}} \Rightarrow \tilde{\mathbb{A}}\mathbb{A} = \mathbb{A}\tilde{\mathbb{A}} = \mathbb{1}$$

$$|\mathbb{A}| = \pm 1 \qquad (|\mathbb{A}| = +1 \text{ når } \mathbb{A} \text{ framgår kontinuerlig fra } \mathbb{1})$$

- Eulervinklene:

$$\varphi \mathrel{\hat{=}} \text{rot. om } z \Rightarrow (\xi, \eta, \zeta)$$
$$\theta \mathrel{\hat{=}} \text{—''— } \xi \Rightarrow (\xi', \eta', \zeta')$$
$$\psi \mathrel{\hat{=}} \text{—''— } \zeta' \Rightarrow (x', y', z')$$

$$\Rightarrow \mathbb{x}' = \mathbb{A}\mathbb{x} = \mathbb{B}\mathbb{C}\mathbb{D}\mathbb{x} \qquad (\mathbb{x} = \mathbb{A}^{-1}\mathbb{x}' = \tilde{\mathbb{A}}\mathbb{x}')$$

med

$$\mathbb{D} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbb{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- infinitesimale rotasjoner: $\mathbb{x}' = (\mathbb{1} + \mathbb{E})\mathbb{x}$; $\mathbb{E}$ antisymm.

$$\Rightarrow d\mathbb{x} \equiv \mathbb{x}' - \mathbb{x} = \mathbb{E}\mathbb{x} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow d\vec{r} = \vec{r} \times d\vec{\Omega} = \vec{r} \times \vec{n}\, d\Phi$$

- tidsendring av vektor: $\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \vec{\omega} \times \vec{G}$; $\vec{\omega} = \frac{d\vec{\Omega}}{dt}$

- krefter i roterende system:

$$\vec{F}_{eff} = m\vec{a}_r$$
$$= \vec{F} + 2m\vec{v}_r \times \vec{\omega} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + m\vec{r} \times \dot{\vec{\omega}}$$

$$\vec{F} = m\vec{a}_s$$

KAP 5: Bevegelseslign. for stive legemer

- $\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$; $I_{jk} = \int_V \rho(\vec{r})\,(r^2\delta_{jk} - x_j x_k)\,dV$
- $T = \frac{1}{2} I \omega^2$; $I = \vec{n}\,\overleftrightarrow{I}\,\vec{n}$
- Hovedakser x_1, x_2, x_3 $\Rightarrow \overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \Rightarrow$ $L_i = I_i \omega_i$, $T = \sum_j \frac{1}{2} I_j \omega_j^2$
- Eulerligningane: $\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}$ (sett fra roterende system)
- Fri rotasjon - presesjon ; Gyroskopeffekt ; Snurrebass

KAP 6: Små oscillasjoner

- små utsving fra stabil likevekt $\Rightarrow V \approx \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j = \frac{1}{2} V_{ij}\,\eta_i \eta_j$
- kin. energi : $T = \frac{1}{2} m_{ij}\,\dot{\eta}_i \dot{\eta}_j = \frac{1}{2} T_{ij}\,\dot{\eta}_i \dot{\eta}_j$
- $L = T - V$; $\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = 0 \Rightarrow T_{ij}\,\ddot{\eta}_j + V_{ij}\,\eta_j = 0$
- $\eta_i = A_i e^{-i\omega t} \Rightarrow \underbrace{\det(\mathbb{V} - \omega^2 \mathbb{T}) = 0}_{\text{sekulær ligning}} \Rightarrow$ egenfrekvenser ω_α
- $A_{i\alpha} \sim \Delta_{i\alpha} \sim (i,\alpha)$-minoren til $|\mathbb{V} - \omega_\alpha^2 \mathbb{T}|$
- $\mathrm{Re}\,\eta_i = \sum_\alpha \mathrm{Re}\,\eta_{i\alpha} = \mathrm{Re} \sum_\alpha C_\alpha \Delta_{i\alpha} e^{-i\omega_\alpha t} = \sum_\alpha \Delta_{i\alpha}\,\Theta_\alpha(t)$

$$\Rightarrow \ldots\ldots \Rightarrow \ddot{\Theta}_\alpha + \omega_\alpha^2\,\Theta_\alpha = 0 \quad \text{(dekoblet!)}$$

Normalkoordinater

- Systemet oscillerer i normalmode α med egenfrekvens ω_α

KAP 9: Kanoniske transformasjoner

- faserommet: $q_i, p_i \Rightarrow 2n$ akser (konfigurasjonsrommet: q_i)

- kanonisk transformasjon: $\{q_i, p_i\} \to \underbrace{\{Q_i, P_i\}}_{\text{nye kanoniske koordinater}}$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad ; \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

$K = K(Q, P, t)$ = Hamiltonfunk. i nye koord.

- modifisert Hamiltons prinsipp $\Rightarrow \underbrace{p_i \dot{q}_i - H}_{L \text{ i "gamle"}} = \underbrace{P_i \dot{Q}_i - K + \frac{dF}{dt}}_{L' \text{ i "nye"}}$

- F = genererende funksjon, bro fra $\{q, p\}$ til $\{Q, P\}$ (4 typer)

- Poissonklammer: $[u, v] = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$

- $\dot{F} = 0 \Rightarrow [F, H] = 0$ (H = Hamiltonfunksjon; forutsetter $\partial F/\partial t = 0$)

- Poissons teorem: $\dot{F} = \dot{G} = 0 \Rightarrow \frac{d}{dt}[F, G] = 0$

- Invarians ved kanonisk transf: $[u, v]_{q,p} = [u, v]_{Q,P}$

KAP 7: S. R.

- c invariant $\Rightarrow x^2+y^2+z^2-c^2t^2$ invariant (under LT)
- 4-vektor: $x_\mu = (\vec{r}, ict)$, dvs $x_4 = ict$ (kompleks metrikk)
- $x_\mu x_\mu$ invariant
- LT = ortogonal transf. i Minkowskirommet
- $\mathbb{x}' = \mathbb{L}\,\mathbb{x}$ med $\mathbb{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{pmatrix}$
- reell metrikk: $ds^2 = g_{\mu\nu}\, dx_\mu\, dx_\nu$; $x_\mu = g_{\mu\nu} x^\nu$; $x^\mu = g^{\mu\nu} x_\nu$

  $\Rightarrow ds^2 = dx_\mu\, dx^\mu$ = invariant

  $x_4 = ict \Rightarrow g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\mathrm{Tr}\, g = 4$, $dx^\mu = dx_\mu$

  $x_0 = ct \Rightarrow g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $\mathrm{Tr}\, g = -2$ eller $g = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $\mathrm{Tr}\, g = +2$

- forminvarians av tensorlign: $C_{\mu\nu} = D_{\mu\nu} \Rightarrow C'_{\mu\nu} = D'_{\mu\nu}$
- egentid τ: $dx_\mu\, dx_\mu = -c^2\, d\tau^2$
- tidsdilatasjon: $dt = \gamma\, d\tau > d\tau$
- romrettet 4-vektor $x_\mu \Rightarrow x_\mu x_\mu > 0$

  tidsrettet ——"——— $\Rightarrow x_\mu x_\mu < 0$

  null ——"——— $\Rightarrow x_\mu x_\mu = 0$

- 4-hastighet: $u_\mu = dx_\mu / d\tau = \gamma(\vec{v}, ic)$; $u_\mu u_\mu = -c^2$
- 4-strømtetthet: $j_\mu = (\vec{j}, ic\rho) = (\rho\vec{v}, ic\rho) = \rho_0 u_\mu$; $\rho = \gamma\rho_0 > \rho_0$
- 4-potensial: $A_\mu = (\vec{A}, i\,\Phi/c)$
- Maxwell: $\Box^2 A_\mu = -\mu_0 j_\mu$; $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \partial_\mu \partial_\mu$
- Lorentzbet: $\partial_\mu A_\mu = 0$ (invariant)
- Kont. lign: $\partial_\mu j_\mu = 0$ (invariant)

- $\frac{d}{d\tau}(m u_\mu) = K_\mu$; $K_\mu = \gamma(\vec{F}, \frac{i}{c}\vec{F}\cdot\vec{v})$
- $d\vec{p}/dt = \vec{F}$; $dE/dt = \vec{F}\cdot\vec{v}$; $p_\mu = (\vec{p}, iE/c)$
- $E^2 = p^2c^2 + m^2c^4$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{pmatrix}$

$F'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} F_{\mu\nu}$

Maxwell: $\partial_\nu F_{\mu\nu} = \mu_0 j_\mu$; $\partial_\nu G_{\mu\nu} = 0$

der $G_{\mu\nu} \triangleq F_{\mu\nu}$ med $-iE_j/c \longleftrightarrow B_j$