Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
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Løsningsforslag til øving 12

Veiledning uke 13

Exercise 1

Kirchhoff's voltage rule (K2) gives

$$\mathcal{E} = \frac{Q}{C} = RI_R$$

while Kirchhoff's current rule (K1) gives

$$I = I_C + I_R$$

Furthermore, we have

$$I_C = \frac{dQ}{dt}$$

Hence:

$$\begin{aligned} I_R(t) &= \frac{V_0}{R}\cos\omega t \\ Q(t) &= V_0 C\,\cos\omega t \\ I_C(t) &= -\omega C V_0\,\sin\omega t = \omega C V_0\,\cos(\omega t + \pi/2) \end{aligned}$$

Total current delivered by the source is therefore

$$I(t) = \frac{V_0}{R}\,\cos\omega t - \omega C V_0\,\sin\omega t$$

We want $I(t)$ on the form

$$I(t) = I_0\,\cos(\omega t - \alpha)$$

with amplitude $I_0 = V_0/Z$, where Z is the impedance of the parallel circuit of R and C, whereas α is the phase constant, i.e., the phase difference between $\mathcal{E}(t)$ and $I(t)$. We have

$$\cos(\omega t - \alpha) = \cos\omega t\,\cos\alpha + \sin\omega t\,\sin\alpha$$

Hence, by direct comparison:

$$\begin{aligned} \frac{\cos\alpha}{Z} &= \frac{1}{R} \\ \frac{\sin\alpha}{Z} &= -\omega C \end{aligned}$$

These two equations, with the unknowns Z and α, are easily solved. We find

$$
\begin{aligned}
Z &= \frac{R}{\sqrt{1+(\omega RC)^2}} \\
I_0 &= \frac{V_0}{R}\sqrt{1+(\omega RC)^2} \\
\alpha &= -\arctan(\omega RC)
\end{aligned}
$$

In the limit $\omega \to 0$ we should find "well known" results from the DC examples in the lectures, and we do: $Z \to R$ and $\alpha \to 0$ so that $I_0 \to V_0/R$. All the current goes through R, whereas the capacitance C now represents an open circuit, transmitting no direct current.

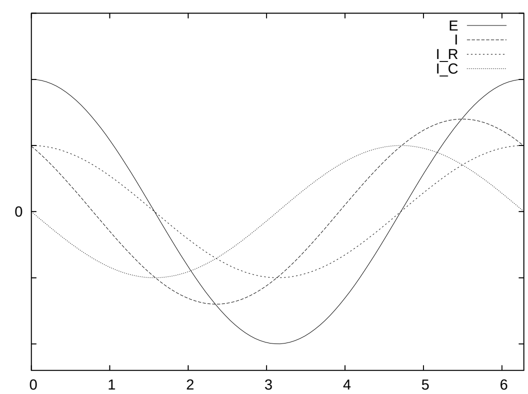
With the given numerical values we have:

$$\omega RC = 2\pi \cdot 10^6 \cdot 10 \cdot 16 \cdot 10^{-9} = 1.0$$

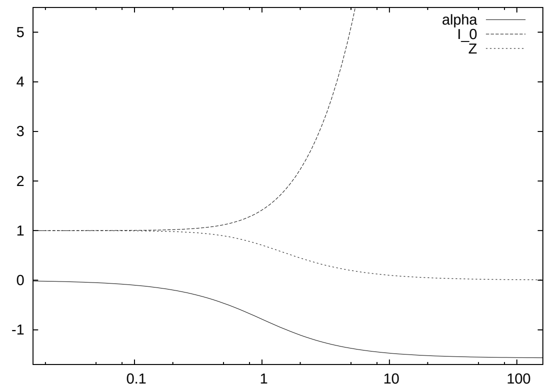
so that

$$
\begin{aligned}
I_0 &= \frac{1.0}{10} \cdot \sqrt{2} = 0.14 \text{ A} \\
\alpha &= -\arctan 1.0 = -45^\circ
\end{aligned}
$$

Sketch of $\mathcal{E}(t)$, $I(t)$, $I_R(t)$ and $I_C(t)$:



Sketch of α, I_0 and Z (with α in radians and ωRC between 0.016 and 160 along the horizontal axis).



Exercise 2

In the first experiment, $B = 0$. Then, Newton's second law is:

$$\begin{aligned} \boldsymbol{F} &= m\boldsymbol{a} = q\boldsymbol{E} \\ \Rightarrow \frac{d\boldsymbol{v}}{dt} &= \frac{q}{m}\boldsymbol{E} \\ \Rightarrow \boldsymbol{v}(t) &= \boldsymbol{v}(0) + \frac{q}{m}\boldsymbol{E}t = \frac{d\boldsymbol{r}}{dt} \\ \Rightarrow \boldsymbol{r}(t) &= \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{q}{2m}\boldsymbol{E}t^2 \end{aligned}$$

Here, it is natural to choose $t = 0$ the moment the particle enters the region with $E \neq 0$, and furthermore, to choose the origin in this position:

$$\boldsymbol{r}(0) = (x_0, y_0) = (0, 0)$$

Here, the velocity is

$$\boldsymbol{v}(0) = v\,\hat{x}$$

when we orient the x axis towards the right. The y axis is oriented upwards, so that

$$\boldsymbol{E} = -E\,\hat{y}$$

(i.e., with $E > 0$) The particle trajectory thus becomes a parabola, just like when we throw a mass in the field of gravity. The velocity in the x direction is not affected by the electric field, so

$$x(t) = vt$$

whereas the particle obtains a constant acceleration in the y direction, i.e., the displacement in the y direction, as a function of t, must be determined by

$$y(t) = -\frac{q}{2m}Et^2$$

The particle will leave the region where $E \neq 0$ at the moment

$$t_L = \frac{x(t_L)}{v} = \frac{L}{v}$$

The vertical position is then

$$y(t_L) = -\frac{q}{2m}E\frac{L^2}{v^2}$$

Already, we may conclude that $q < 0$ if $y(t_L) > 0$.
The distance from $x = L$ to $x = L + D$ is then traveled without influence from any kind of forces, and with a direction relative to the x axis in terms of the angle α, where

$$\tan\alpha = \frac{v_y(t_L)}{v_x(t_L)} = \frac{-\frac{q}{m}E\frac{L}{v}}{v} = -\frac{qEL}{mv^2}$$

Besides, we must have

$$\tan\alpha = \frac{y - y(t_L)}{D}$$

where y is where the electron hits the detector, at $x = L + D$.

The experiment is then repeated with the same E-field, but now we turn on a magnetic field B directed into the plane, so that the particles are no longer deflected by the fields. This implies that the electric force (upwards) is exactly balanced by the magnetic force (downwards). In other words:

$$\begin{aligned} \boldsymbol{F} &= q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B} = 0 \\ \Rightarrow E &= vB \\ \Rightarrow \frac{1}{v} &= \frac{B}{E} \end{aligned}$$

Hence:

$$\begin{aligned} \frac{y - y(t_L)}{D} &= -\frac{qEL}{mv^2} = -\frac{qEL}{m}\cdot\frac{B^2}{E^2} \\ \Rightarrow y + \frac{q}{2m}EL^2\frac{B^2}{E^2} &= -\frac{qEL}{m}\cdot\frac{B^2}{E^2}D \\ \Rightarrow yE &= -\frac{q}{m}\cdot B^2\left(DL + \frac{1}{2}L^2\right) \\ \Rightarrow \frac{q}{m} &= -\frac{yE}{B^2\left(DL + \frac{1}{2}L^2\right)} \end{aligned}$$

I.e.,

$$a = \frac{E}{B^2\left(DL + L^2/2\right)}$$

Exercise 3

a) The speed of the ions when they enter the region with magnetic field is determined by the change in potential energy, going through the voltage difference V, being equal to the change in the kinetic energy of the ions:

$$eV = \frac{1}{2}mv^2 \quad\Rightarrow\quad v = \sqrt{\frac{2eV}{m}}$$

The centripetal acceleration inside the magnetic field is

$$a = \frac{v^2}{r}$$

so that Newton's 2. law gives

$$F = m\frac{v^2}{r} = evB \quad \Rightarrow \quad r = \frac{mv}{eB}$$

Radius for the resulting circular path for a particle with mass m becomes

$$r = \frac{1}{B}\sqrt{\frac{2Vm}{e}}$$

i.e., proportional with $\sqrt{m}$. Radii and masses for the different isotopes must be related as follows:

$$\frac{r_i}{r_j} = \sqrt{\frac{m_i}{m_j}}$$

where $i, j = 79$ or 81.
If the points where the ions hit the photographic plate are supposed to be separated by a distance of (at least) $a = 1.0$ cm, the *diameter* of the two circular paths must differ by 1.0 cm. We obtain

$$a = 1.0 \text{ cm } = 2(r_{81} - r_{79}) = 2r_{79}\left(\sqrt{\frac{m_{81}}{m_{79}}} - 1\right)$$

This gives

$$r_{79} = \frac{a}{2}\left(\sqrt{\frac{m_{81}}{m_{79}}} - 1\right)^{-1} = 0.5 \text{ cm } \cdot \left(\sqrt{\frac{81}{79}} - 1\right)^{-1} \simeq 39.7 \text{ cm}$$

and

$$r_{81} = r_{79} + \frac{a}{2} \simeq 40.2 \text{ cm}$$

Now, we can determine how strong magnetic field that can be used to achieve these radii:

$$B = \frac{1}{r_{81}}\sqrt{\frac{2Vm_{81}}{e}} = \frac{1}{0.402} \cdot \sqrt{\frac{2 \cdot 400 \cdot 81 \cdot 1.67 \cdot 10^{-27}}{1.6 \cdot 10^{-19}}} = 0.065 \text{ T}$$

This represents the upper limit of B: A stronger magnetic field will reduce both r_{79} and r_{81}, but r_{81} the most, so that the "hit points" move closer to each other. However, at the same time the diameter $d_{81} = 2r_{81}$ must not be larger than the physical limit of the instrument, given by $L = 250cm$. That corresponds to a minimum value of the magnetic field strength:

$$B_{\text{min}} = \frac{1}{L/2}\sqrt{\frac{2Vm_{81}}{e}} = \frac{1}{1.25} \cdot \sqrt{\frac{2 \cdot 400 \cdot 81 \cdot 1.67 \cdot 10^{-27}}{1.6 \cdot 10^{-19}}} = 0.021 \text{ T}$$

In other words, we may use a magnetic field between 21 and 65 mT.