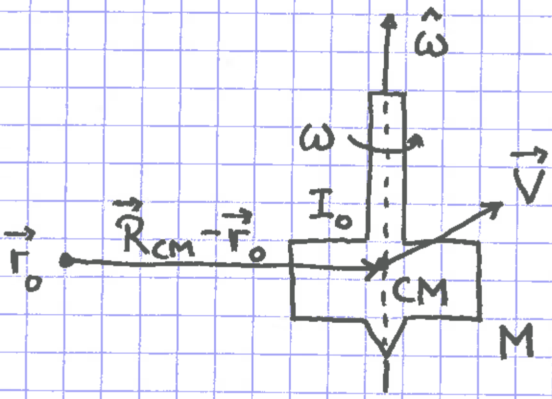
22.09.14

$\vec{L}$ for stivt legeme [YF 10.5; LL 6.6]

Forbehold: Legeme med ~~sylinder~~refleksjons-symmetri om rotasjonsaksen.



Fra s. 36: $K = K_{trans} + K_{rot} = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2$

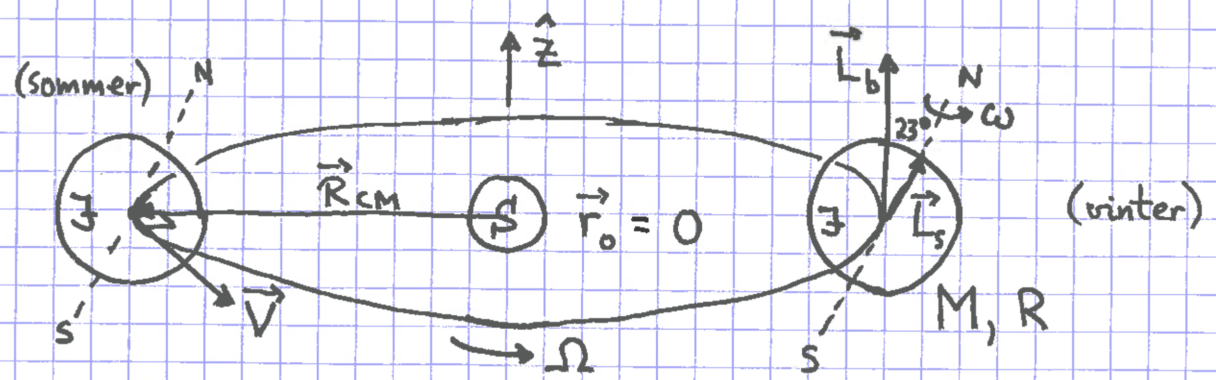
Tilsvarende: $\boxed{\vec{L} = \vec{L}_b + \vec{L}_s = M(\vec{R}_{CM}-\vec{r}_0)\times\vec{V} + I_0\vec{\omega}}$

[Bevis: s. 47A, 47B]

Banedreieimpuls, relativt $\vec{r}_0$: $\vec{L}_b = M(\vec{R}_{CM}-\vec{r}_0)\times\vec{V}$

Indre dreieimpuls, spinn, uavh. av $\vec{r}_0$: $\vec{L}_s = I_0\vec{\omega}$

Eks 1: $\vec{L}$ til Jorda (J) relativt Sola (S)



$$\vec{L} = \vec{L}_b + \vec{L}_s = \vec{R}_{CM}\times M\vec{V} + I_0\vec{\omega} = R_{CM}MV\hat{z} + I_0\vec{\omega}$$

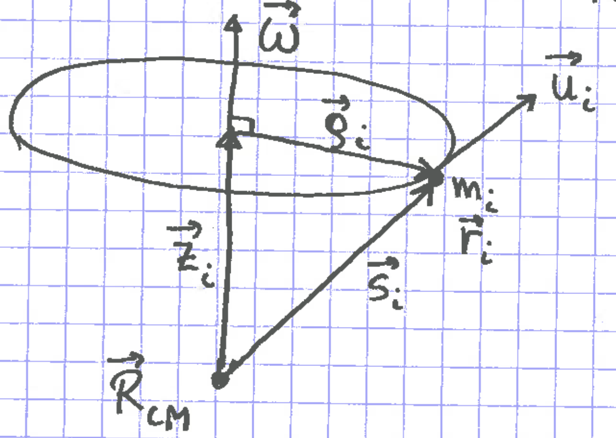
$R_{CM} \sim 1.5\cdot 10^{11}$ m, $M \sim 6\cdot 10^{24}$ kg, $V = R_{CM}\,\Omega$; $\Omega = 2\pi/1$ år

$I_0 \sim \frac{1}{3}MR^2$; $R \sim 6.37\cdot 10^6$ m, $\omega = 2\pi/1$døgn

$\Rightarrow L_b \sim 2.7\cdot 10^{40}$ Js ; $L_s \sim 6\cdot 10^{33}$ Js $\Rightarrow L_b \gg L_s$

Bevis for $\vec{L} = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + I_0 \vec{\omega}$

for stivt legeme med ~~sylinder~~ refleksjons-symmetri om rotasjonsaksen



$$\vec{r}_i = \vec{R}_{CM} + \vec{s}_i$$

$$\vec{v}_i = \vec{V} + \vec{u}_i$$

$$\vec{s}_i = \vec{z}_i + \vec{\rho}_i$$

$$\vec{u}_i = \vec{\omega} \times \vec{\rho}_i = \vec{\omega} \times \vec{s}_i \qquad (\text{siden } \vec{\omega} \times \vec{z}_i = 0)$$

$$\begin{aligned}
\vec{L} &= \sum_i m_i (\vec{r}_i - \vec{r}_0) \times \vec{v}_i \\
&= \sum_i m_i (\vec{R}_{CM} - \vec{r}_0 + \vec{s}_i) \times (\vec{V} + \vec{u}_i) \\
&= \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + \sum_i m_i \vec{s}_i \times \vec{V} \\
&\quad + \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{u}_i + \sum_i m_i \vec{s}_i \times \vec{u}_i
\end{aligned}$$

1. sum: $\sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{V} = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} = \vec{L}_b$

2. sum: $\sum_i m_i \vec{s}_i \times \vec{V} = \sum_i m_i (\vec{r}_i - \vec{R}_{CM}) \times V = (M\vec{R}_{CM} - M\vec{R}_{CM}) \times \vec{V} = 0$

3. sum: $\sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times \vec{u}_i = \sum_i m_i (\vec{R}_{CM} - \vec{r}_0) \times (\vec{\omega} \times \vec{s}_i)$

$$= (\vec{R}_{CM} - \vec{r}_0) \times (\vec{\omega} \times \underbrace{\sum_i m_i \vec{s}_i}_{=0}) = 0$$

4. sum: $\sum_i m_i \vec{s}_i \times \vec{u}_i =$ dreieimpuls pga masseelementenes bevegelse relativt CM

(Til nå: <u>Helt</u> generelt!)

Stivt legeme: $\vec{u}_i = \vec{\omega} \times \vec{s}_i$

$$\Rightarrow \sum_i m_i \vec{s}_i \times \vec{u}_i = \sum_i m_i \vec{s}_i \times (\vec{\omega} \times \vec{s}_i)$$

Vektoridentitet: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A} \cdot \vec{C}) - \vec{C}\,(\vec{A} \cdot \vec{B})$

$$\Rightarrow \sum_i m_i \vec{s}_i \times (\vec{\omega} \times \vec{s}_i) = \sum_i m_i \left\{ \vec{\omega}\, s_i^2 - \vec{s}_i\,(\vec{s}_i \cdot \vec{\omega}) \right\}$$

$$= \sum_i m_i \left\{ \vec{\omega}\,(z_i^2 + \rho_i^2) - (\vec{z}_i + \vec{\rho}_i)\, z_i \omega \right\}$$

$$= \sum_i m_i \left\{ \vec{\omega}\,(z_i^2 + \rho_i^2) - z_i^2 \vec{\omega} - z_i \omega \vec{\rho}_i \right\}$$

$$= \sum_i m_i \rho_i^2 \vec{\omega} - \omega \sum_i m_i z_i \vec{\rho}_i$$

$$= \underbrace{I_0 \vec{\omega}}_{\vec{L}_s} - \omega \sum_i m_i z_i \vec{\rho}_i$$

Med refleksjons- ~~spekter~~symmetri om $\hat{\omega}$, dvs om $\hat{z}$:

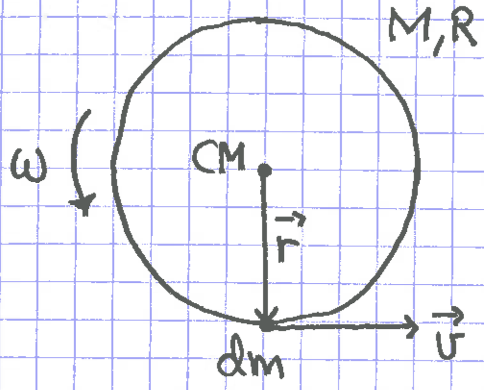
$$\sum_i m_i z_i \vec{\rho}_i = \sum_i m_i z_i (x_i \hat{x} + y_i \hat{y}) = 0$$

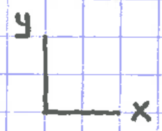
fordi bidraget fra (x_i, y_i) kanselleres av bidraget fra $(-x_i, -y_i)$!

Dermed:

$$\vec{L} = \vec{L}_b + \vec{L}_s = M(\vec{R}_{CM} - \vec{r}_0) \times \vec{V} + I_0 \vec{\omega} \qquad \text{qed}$$

Eks 2: Ring; ren rotasjon om CM.





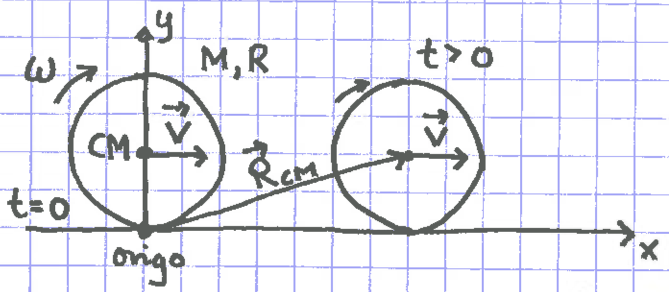
$$\vec{R}_{CM} = \vec{r}_0 = 0 \quad \text{(velges)}$$
$$\vec{\omega} = \omega\hat{z}$$

$$d\vec{L} = \vec{r} \times d\vec{p} = \vec{r} \times dm \cdot \vec{v}$$
$$= R \cdot dm \cdot \omega R \cdot \hat{z}$$

$$\Rightarrow \vec{L} = \left(\int dm\right) R^2 \omega \hat{z} = MR^2 \omega \hat{z} = \underline{\underline{I_0 \vec{\omega}}}$$

Dvs: $\vec{L} = \vec{L}_s$; $\vec{L}_b = M\underbrace{(\vec{R}_{CM} - \vec{r}_0)}_{=0} \times \underbrace{\vec{V}}_{=0} = 0$

Eks 3: Rullende ring $(V = \omega R)$



(a) $\vec{r}_0 = \vec{R}_{CM}$; $\vec{L}_{CM} = ?$

(b) $\vec{r}_0 = 0$; $\vec{L}_0 = ?$

Løsn:

(a) Som over, med $\vec{\omega} = -\omega\hat{z}$ $\Rightarrow$ $\vec{L}_{CM} = I_0\vec{\omega} = \underline{\underline{-MR^2\omega\hat{z}}}$

(b) $$\vec{L}_0 = M\vec{R}_{CM} \times \vec{V} + I_0\vec{\omega}$$
$$= -MRV\hat{z} - MR^2\omega\hat{z}$$
$$= \underline{\underline{-2MR^2\omega\hat{z}}}$$

Bevaringslover, oppsummert

- For <u>isolert</u> system (dvs ingen ytre krefter) er total energi, impuls og dreieimpuls bevart.
- Mekanisk energi $E = K + U$ er bevart for <u>konservativt</u> system (dvs ingen <u>dissipasjon</u> pga friksjonsarbeid etc)
- Impuls $\vec{p}$ er bevart for system som ikke påvirkes av netto ytre kraft $\vec{F}$:

  N2, $\vec{F} = d\vec{p}/dt \Rightarrow d\vec{p} = 0$ hvis $\vec{F} = 0$
- Dreieimpuls $\vec{L}$ er bevart for system som ikke påvirkes av netto ytre dreiemoment $\vec{\tau}$:

  N2, rotasjon, $\vec{\tau} = d\vec{L}/dt \Rightarrow d\vec{L} = 0$ hvis $\vec{\tau} = 0$

Mekanisk likevekt [YF 11.1–11.3; LL 7.1]

Et stivt legeme er i ro,

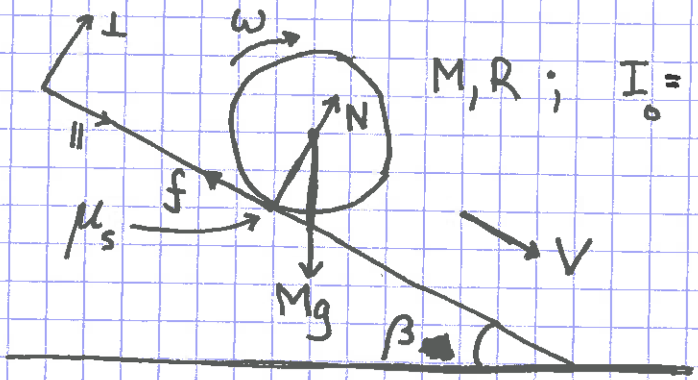
$$\vec{p} = 0 \quad \text{og} \quad \vec{L} = 0,$$

bare dersom

$$\underbrace{\sum_i \vec{F}_i = 0}_{\text{Netto ytre kraft}} \quad \text{og} \quad \underbrace{\sum_i \vec{\tau}_i = 0}_{\text{Netto ytre dreiemoment}}$$

Rotasjonsdynamikk, eksempler

Eks 1: Rulling på skråplan [YF 10.3; LL 6.8]



$M, R\ ;\quad I_0 = c \cdot MR^2$

For ren rulling ($V = \omega R$), bestem $\dot{V}$, og minste tillatte verdi for μ_s.

Løsning:
Ser at V og ω må øke; må ha dreiemoment τ om CM i tråd med det. Kun friksjonskrafta f har "arm" mhp CM. Da må f ha retning oppover skråplanet for å gi $\dot{\omega} > 0$.

N2, ∥: $\quad Mg \sin\beta - f = M\dot{V}$

N1, ⊥: $\quad N = Mg\cos\beta$

N2, rot. om CM: $\quad f \cdot R = I_0 \dot{\omega} = cMR^2 \cdot \frac{\dot{V}}{R} = cMR\dot{V}$

$$f = cM\dot{V}$$

$\Rightarrow \quad Mg\sin\beta - cM\dot{V} = M\dot{V} \quad \Rightarrow \quad \underline{\underline{\dot{V} = g \cdot \frac{\sin\beta}{1+c}}}$

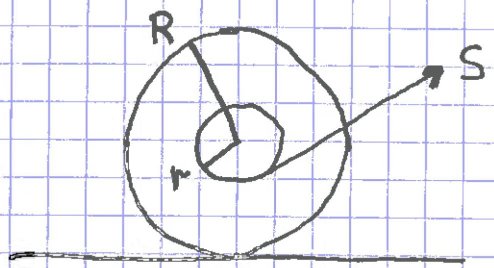
$$f \leq f_{max} = \mu_s N = \mu_s Mg\cos\beta$$

$\Rightarrow \quad cMg\sin\beta/(1+c) \leq \mu_s Mg\cos\beta \quad \Rightarrow \quad \underline{\underline{\mu_s \geq \frac{c}{1+c}\tan\beta}}$

Ring, Sylinderskall: $c = 1 \Rightarrow \dot{V} = \frac{1}{2} g\sin\beta,\quad \mu_s^{min} = \frac{1}{2}\tan\beta$

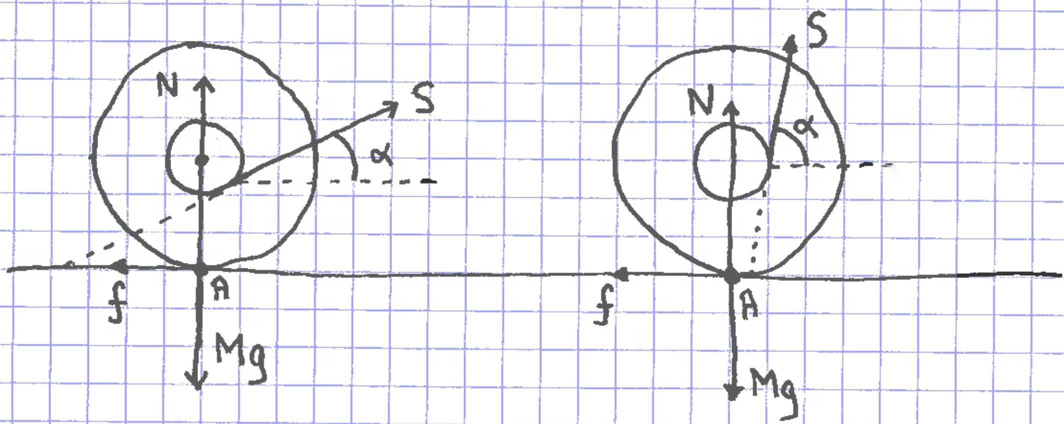
Kompakt kule: $c = 2/5 \Rightarrow \dot{V} = \frac{5}{7} g\sin\beta,\quad \mu_s^{min} = \frac{2}{7}\tan\beta$

Eks 2: Snelle, Jojo



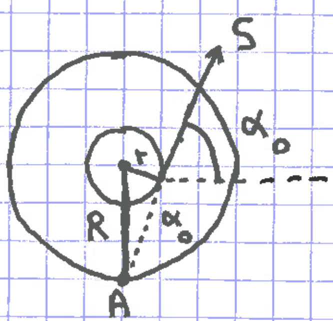
Hvilken vei ruller snella?
(Anta ren rulling.)

Løsning:



N2, rot. om kontaktpunktet A: Kun snordraget S har en arm mhp A. Liten α $\Rightarrow$ Rot. med klokka $\Rightarrow$ Ruller mot høyre. Stor α $\Rightarrow$ Rot. mot klokka $\Rightarrow$ Ruller mot venstre.

Hvilken retning på S gir statisk likevekt?

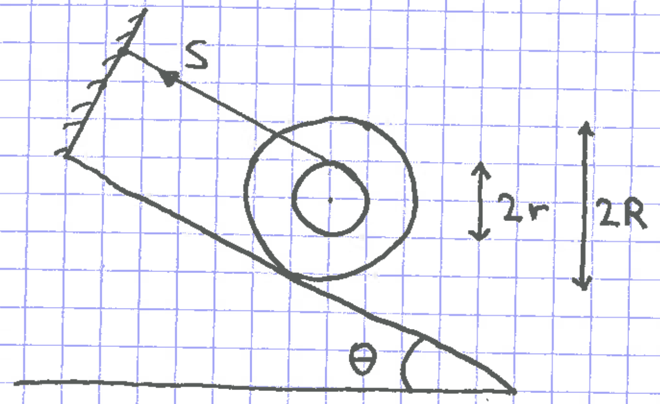


Ingen arm mhp A for S når $\alpha = \alpha_0$, gitt ved

$$\underline{\underline{\cos\alpha_0 = \frac{r}{R}}}$$

[Kan selvsagt ikke trekke for hardt! Med $\alpha = \alpha_0$ må S være mindre enn $\mu_s Mg \,/\, \left(\frac{r}{R} + \mu_s\sqrt{1 - r^2/R^2}\right)$ for at snella skal ligge i ro. Vis dette!]

Eks 3: Sluresnelle (demo + øving)

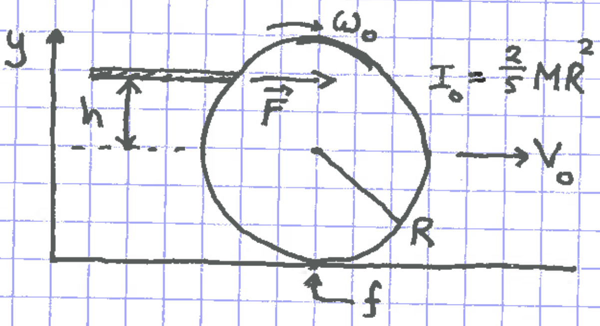


Finn θ_0 = max vinkel uten at snella glir (slurer) nedover skråplanet.
Hva er snordraget S da?

Tips: N1, ∥ ; N1, rot. om CM ; $f = f_{max} = \mu_s N$ når $\theta = \theta_0$

Finn snellas akselerasjon $a = \dot{v} = r\dot{\omega}$ når $\theta > \theta_0$.
Tips: N2, ∥ ; N2, rot. om CM ; $f = \mu_k N$

Eks 4: Snooker (øving) [LL 6.7]



Kort støt:

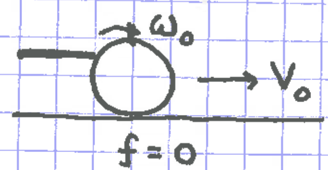
$F \cdot \Delta t = \Delta p = MV_0$ (N2)

$\left.\begin{array}{l} \tau \cdot \Delta t = \Delta L = I_0 \omega_0 \\ \tau = F \cdot h \end{array}\right\}$ (N2, rot. om CM)

($F \gg f$ i støtet)

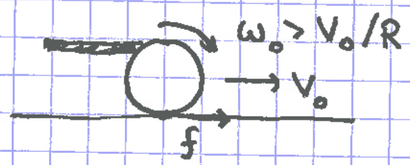
Etter støtet:

Ren rulling hvis $h = h_0$, $V_0 = \omega_0 R$:



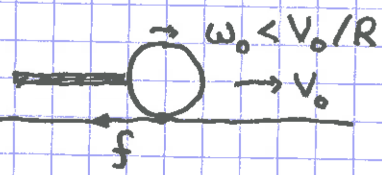
$h_0 = ?$

Hvis $h > h_0$:



Sluring

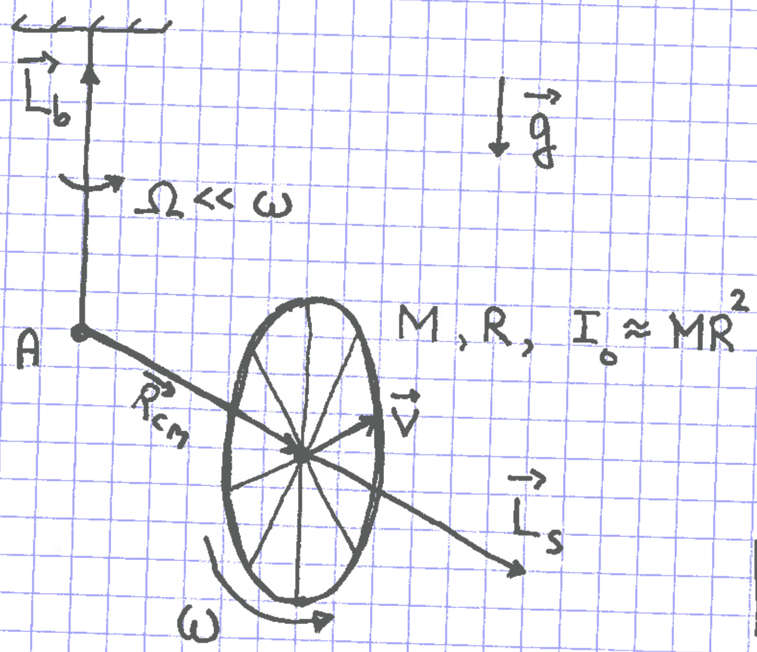
Hvis $h < h_0$:



Sluring

} Etter hvert: Ren rulling!

Eks 5: Presesjon (demo) [YF 10.7; LL 6.10]



Målte verdier:

$M \approx 5$ kg

$R \approx 0.3$ m

$R_{CM} \approx 0.2$ m

$T_\Omega = \frac{2\pi}{\Omega} \approx 4.25$ s

$\boxed{\text{Beregn } \omega}$

Løsning: A = valgt ref. punkt

$$\vec{L}_A = \vec{L}_b + \vec{L}_s = \vec{R}_{CM} \times M\vec{V} + I_0\vec{\omega}$$

$$L_b = R_{CM} M V \ll L_s = MR^2\omega$$

$$(\vec{L}_b \approx \text{konst.})$$

Tyngden $M\vec{g}$ har arm $\vec{R}_{CM}$ relativt A, og dermed dreiemomentet

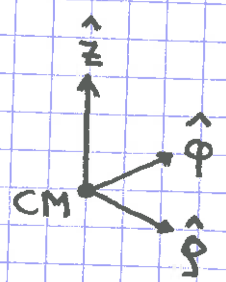
$$\vec{\tau}_A = \vec{R}_{CM} \times M\vec{g}$$

med abs.verdi

$$\tau_A = R_{CM} M g$$

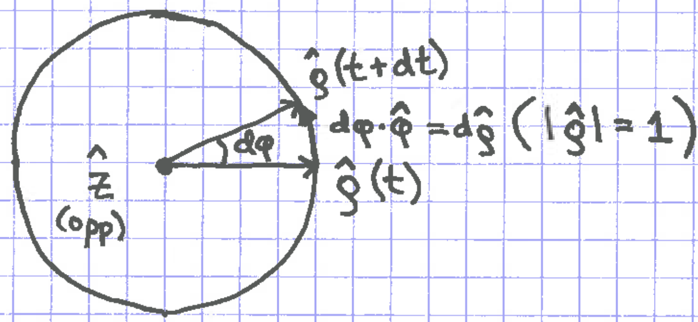
og retning som $\vec{V}$, dvs

$$\vec{\tau}_A = R_{CM} M g \,\hat{\varphi}$$



N2, rot. mhp A: $\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = \frac{d\vec{L}_s}{dt} = I_0 \frac{d\vec{\omega}}{dt}$

$\vec{\omega}(t) = \omega \cdot \hat{\rho}(t)$



$$\frac{d\vec{\omega}}{dt} = \omega \frac{d\hat{\rho}}{dt} = \omega \frac{d\varphi}{dt} \hat{\varphi}$$

$$= \omega \Omega \hat{\varphi}$$

Dermed: $R_{CM} Mg = I_0 \omega \Omega = MR^2 \omega \Omega$

$$\Rightarrow \quad \omega = \frac{R_{CM}\, g}{R^2 \, \Omega}$$

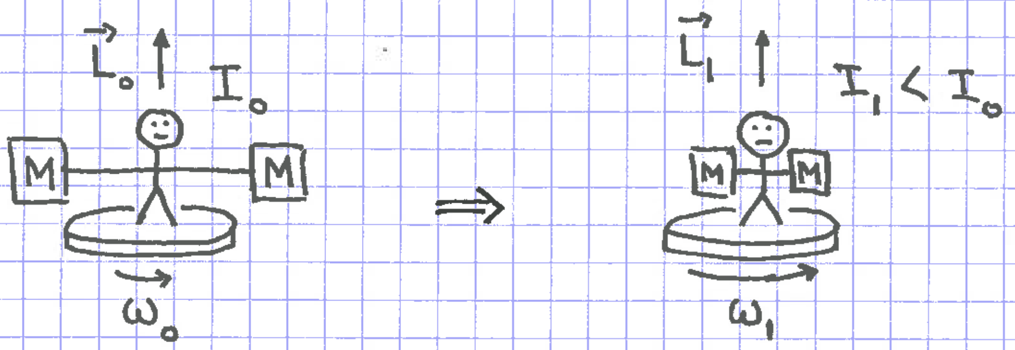
$$\Rightarrow \quad \underline{\underline{T_\omega = \frac{2\pi}{\omega} = \frac{(2\pi R)^2}{R_{CM}\, g \, T_\Omega}}}$$

Tallverdier ($g \approx 10 \text{ m/s}^2$):

$$T_\omega = \frac{(2\pi \cdot 0.3)^2}{0.2 \cdot 10 \cdot 4.25} \text{ s} \approx \underline{0.42 \text{ s}}$$

Høres vel rimelig ut !? (Ca to og en halv omdreining pr sekund.)

Eks 6 : Piruett (demo)



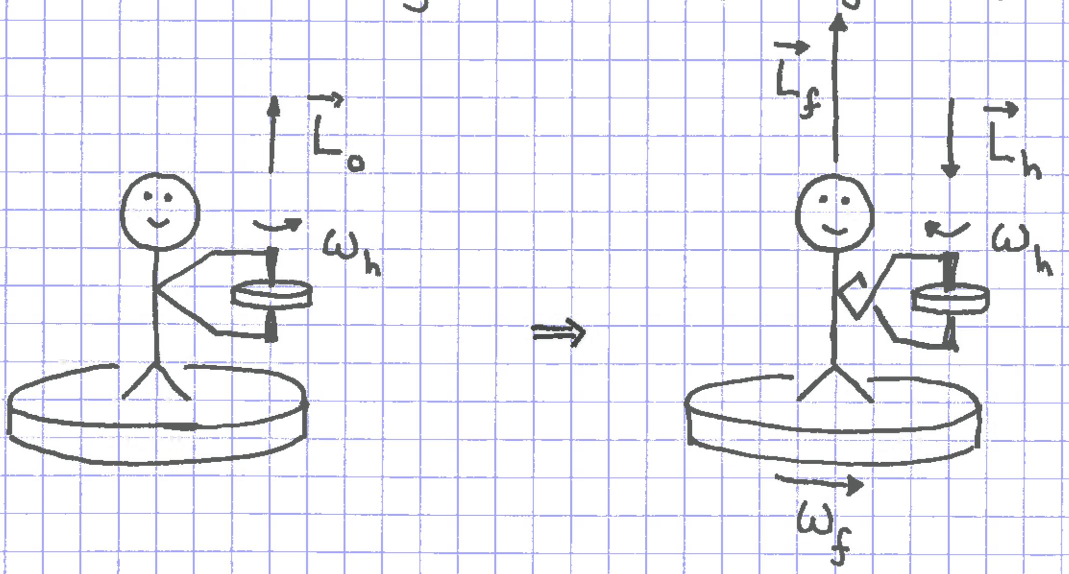
$\tau_{ytre} = 0 \Rightarrow \vec{L}$ er bevart $\Rightarrow \vec{L}_0 = \vec{L}_1$

$$\Rightarrow I_0 \omega_0 = I_1 \omega_1 \Rightarrow \omega_1 = \frac{I_0}{I_1}\omega_0 > \omega_0$$

Vis at mekanisk energi <u>øker</u> : $K_1 = \frac{1}{2} I_1 \omega_1^2 > \frac{1}{2} I_0 \omega_0^2 = K_0$

Forklar hvorfor !

Eks 7 : Bevaring av $\vec{L}$ for hjul + foreleser (demo)
(+stol)



h: hjul
f: foreleser

$\tau_{ytre} = 0 \Rightarrow \Delta \vec{L} = 0$

$$\Rightarrow \vec{L}_f + \vec{L}_h = \vec{L}_0$$

$$|\vec{L}_0| = |\vec{L}_h| \; ; \quad \vec{L}_f = 2\vec{L}_0$$