Institutt for fysikk, NTNU
TFY4155/FY1303 Elektrisitet og magnetisme
Vår 2004

Solution to øving 5

Guidance Monday February 9

Exercise 1

a) Here, $V(r,\theta)$ and the gradient operator are given in spherical coordinates, so it's simply a matter of performing a couple of differentiations (nothing depends upon the angle ϕ, so we need not care about the term containing $\partial/\partial\phi$):

$$\begin{aligned}\boldsymbol{E}(r,\theta) &= -\nabla V(r,\theta) \\ &= -\left(\hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \\ &= \hat{r}\frac{p\cos\theta}{2\pi\varepsilon_0 r^3}+\hat{\theta}\frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\end{aligned}$$

Note that the electric field from an electric dipole, at large distance r from the dipole, falls off as $1/r^3$, i.e., faster than the field from an electric "monopole", i.e., a point charge, where the field falls off as $1/r^2$. The contributions to the total field from the two point charges enter with opposite sign and thereby cancel each other partly. (However, they do not cancel completely, since the directions of the two contributions to $\boldsymbol{E}$ will always be different.)

With $\theta = 0$, we have

$$E_r = \frac{p}{2\pi\varepsilon_0 r^3}$$

and

$$E_\theta = 0$$

This is reasonable: We are now in a point far out on the z axis, so that the radial direction is exactly along the z axis, while the θ direction will be along the x axis. And on the z axis, the electric field is obviously directed along the z axis.
With $\theta = \pi/2$, we have

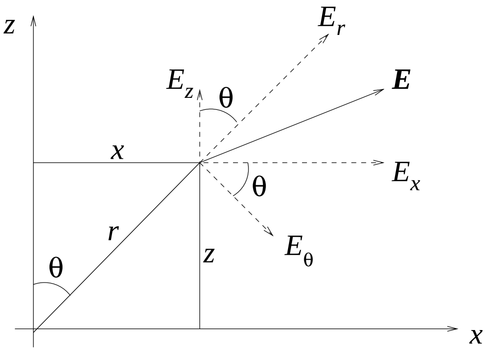
$$E_r = 0$$

and

$$E_\theta = \frac{p}{4\pi\varepsilon_0 r^3}$$

This is also reasonable: We are now in a point far out on the x axis, so that the radial direction is exactly along the x axis, while the θ direction will be along the negative z axis. And on the x axis, the electric field is obviously directed along the negative z axis.
If we insert $r = 0$ into the expressions for E_r and E_θ, we see that they both become infinite. However, this is not a real problem, because now we are trying to use an expression for E outside its region of validity $r \gg a$. The field in the origin is certainly not infinite, and also not very difficult to calculate. I'm sure you manage to do that by yourself!

b) From the figure below, it should be relatively clear that the electric field $\boldsymbol{E}$ may be decomposed either in terms of E_r and E_θ, or in terms of E_x and E_z.



Both E_r and E_θ have components in the x direction, and the total x component of the field must be the sum of these two. Looking at the figure, we have

$$\begin{aligned} E_x &= E_r \sin\theta + E_\theta \cos\theta \\ &= \frac{p\cos\theta}{2\pi\varepsilon_0 r^3} \cdot \sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \cdot \cos\theta \\ &= \frac{3p}{4\pi\varepsilon_0 r^3} \sin\theta\cos\theta \\ &= \frac{3pxz}{4\pi\varepsilon_0 r^5} \\ &= \frac{3pxz}{4\pi\varepsilon_0 \left(x^2+z^2\right)^{5/2}} \end{aligned}$$

Here, we have used $\sin\theta = x/r$, $\cos\theta = z/r$ and $r = (x^2+z^2)^{1/2}$.
Likewise, we find the z component:

$$\begin{aligned} E_z &= E_r \cos\theta - E_\theta \sin\theta \\ &= \frac{p\cos\theta}{2\pi\varepsilon_0 r^3} \cdot \cos\theta - \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \cdot \sin\theta \\ &= \frac{p}{4\pi\varepsilon_0 r^3} \left(2\cos^2\theta - \sin^2\theta\right) \\ &= \frac{\left(2z^2 - x^2\right)p}{4\pi\varepsilon_0 \left(x^2+z^2\right)^{5/2}} \end{aligned}$$

c) In cartesian coordinates, the potential becomes

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p(z/r)}{4\pi\varepsilon_0 r^2} = \frac{pz}{4\pi\varepsilon_0 \left(x^2+z^2\right)^{3/2}}$$

With $\boldsymbol{E} = -\nabla V$ we then have

$$E_x = -\frac{\partial V}{\partial x} = \frac{3pzx}{4\pi\varepsilon_0 \left(x^2+z^2\right)^{5/2}}$$

$$
\begin{aligned}
E_z &= -\frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left(\frac{-p}{(x^2+z^2)^{3/2}} + \frac{3pz^2}{(x^2+z^2)^{5/2}}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\left(\frac{-p\,(x^2+z^2)+3pz^2}{(x^2+z^2)^{5/2}}\right) \\
&= \frac{(2z^2-x^2)\,p}{4\pi\varepsilon_0\,(x^2+z^2)^{5/2}}
\end{aligned}
$$

The same as found in *b*)!

Exercise 2
This exercise is about using Gauss' law,

$$\phi = \oint_S \boldsymbol{E}\cdot d\boldsymbol{A} = Q/\varepsilon_0.$$

First thing to notice is that $\boldsymbol{E} = E_x\hat{x}$ in the first three cases is directed along the x axis. That implies that no electric flux passes through the faces of the cube having their surface normal in the y or in the z direction. We only have nonzero flux through the surfaces with surface normal in the x direction. In *d*), we must also include the two surfaces with surface normal in the y direction.

a) Here, $E_x = C$, i.e., constant. That means we have the same amount of electric flux in through the surface at $x = 0$ as we have out through the surface at $x = a$:

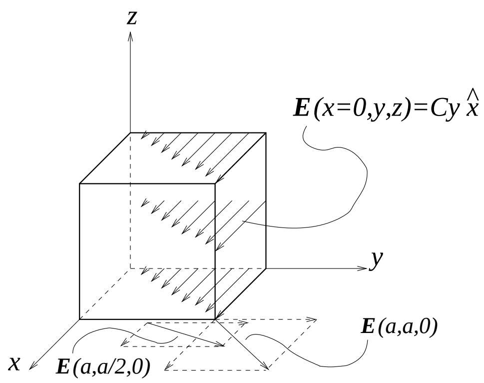
$$\phi_{\text{inn}} = \phi_{\text{ut}} = E_x A = Ca^2$$

where $A = a^2$ is the area of one of the faces of the cube. The *net* flux becomes $\phi = 0$ through the whole surface of the cube, and accordning to Gauss' law, the net charge inside the cube is also zero, $Q = 0$.

b) Here, $E_x = Cx$. On the two faces at $x = 0$ and $x = a$, the electric field strength is then $E_x(0) = 0$ and $E_x(a) = Ca$. Thus, we have zero flux through the face at $x = 0$, whereas the flux out at $x = a$ becomes $\phi_{\text{ut}}(x = a) = Ca \cdot a^2 = Ca^3$. The latter also becomes the net flux through S, $\phi = Ca^3$. Gauss' law tells us that the net charge inside the cube is now $Q = C\varepsilon_0 a^3$.

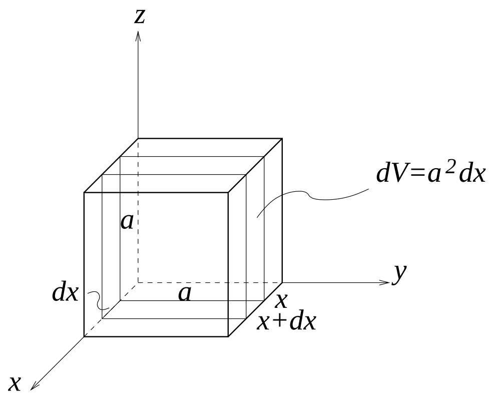
c) Here, $E_x = Cx^2$. Once more, $E_x(0) = 0$, while $E_x(a) = Ca^2$. Again, we have zero flux through the face at $x = 0$, whereas the flux out at $x = a$ becomes $\phi_{\text{ut}}(x = a) = Ca^2 \cdot a^2 = Ca^4$, which also becomes the net flux through S. Using Gauss' law: $Q = C\varepsilon_0 a^4$ is the net charge inside the cube.

d) Now, $E_x = Cy$ and $E_y = Cx$, and we have nonzero flux through four of the six faces of the cube. The figure below illustrates the field on the plane at $x = 0$. In addition, we have included a couple of examples of what the field looks like on the plane at $x = a$, where the y component of the field is constant, equal to Ca, whereas the x component grow linearly with y.



It may seem somewhat complicated at first sight, but simplifies considerably upon closer examination: We notice that E_x does not depend upon x and correspondingly that E_y does not depend on y. This means that the flux in through the face at $x = 0$ must be the same as the flux out through the face at $x = a$. And analogously: Equal flux in through the face at $y = 0$ as flux out at $y = a$. In conclusion: Zero net flux through the whole closed surface, and zero net charge inside the cube, according to Gauss' law.

e) For case *c*), i.e., with $\boldsymbol{E} = Cx^2\hat{x}$, we are supposed to determine the charge density ρ inside the cube.
We follow the hints given in the text and start by looking at a small volume element $dV = a^2\ dx$, i.e., a thin slice located between x and $x + dx$:



To find the net flux out through the surface of this thin slice, we must determine the flux *in* at x and the flux *out* at $x + dx$. The electric field on these two faces is:

$$\begin{aligned}
E(x) &= Cx^2 \\
E(x+dx) &= C\,(x+dx)^2 \\
&= Cx^2 + 2Cx\ dx + C(dx)^2 \\
&\simeq Cx^2 + 2Cx\ dx
\end{aligned}$$

Notice the approximation we do here: We neglect $C(dx)^2$ relative to $2Cx\ dx$. We can do this because dx is an infinitesimal thickness, i.e., we will eventually take the limit $dx \to 0$.
The flux through the two faces is obtained by multiplying the field strength with the surface area, which is a^2. Hence, the net flux out through the small (infinitesimal!) gaussian surface

becomes

$$\phi = E(x + dx)a^2 - E(x)a^2 = 2Ca^2x\ dx$$

According to Gauss' law, this equals the net charge within the slice, divided by ε_0. The net charge within the slice may be written as the *charge density* ρ multiplied with the volume:

$$dq = \rho\ dV = \rho a^2\ dx$$

Thus:

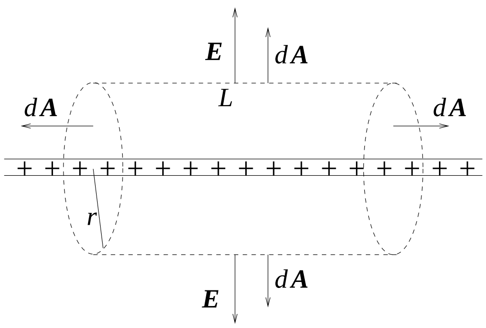
$$\begin{aligned} 2Ca^2x\ dx &= \frac{\rho a^2\ dx}{\varepsilon_0} \\ \Rightarrow \rho &= 2C\varepsilon_0 x \end{aligned}$$

Comments:

- Note that we have here used Gauss' law with a gaussian surface enclosing an infinitesimal ("differential") volume element. That's no problem; it is up to us to choose a sensible gaussian surface - be it large or small. But remember, it must always be a *closed* surface!
- In *c*) above, you found the total charge Q inside the cube. Now, you have determined the charge density ρ. Check that the two answers are consistent with each other!

Exercise 3

An infinitely long, uniformly charged rod must result in an electric field directed radially away from the rod (alternatively, radially towards the rod, if it were negatively charged). Secondly, the electric field strength can only depend upon the distance r from the rod. Thinking for a while, we hopefully realize that a smart choice for a gaussian surface is a cylinder, with the charged rod passing through the center axis of the cylinder:



On the surface of this cylinder, the "surface element vector" $d\boldsymbol{A}$ will then either be perpendicular to $\boldsymbol{E}$ (on the two flat end faces of the cylinder) or parallel to $\boldsymbol{E}$ (on the curved part of the cylinder surface). On this curved part, i.e., where we get a nonzero contribution to the surface integral in Gauss' low, $E(r)$ has everywhere the same value. Hence, it may be taken outside the integral. On the two flat ends, $\boldsymbol{E}$ is parallel to the surface (i.e., normal to the surface normal!), so there is no electric flux passing through these parts of the cylinder surface. The circumference of the cylinder is $2\pi r$ and its length is L. Therefore, the area of the curved part

is $2\pi r L$. How much charge is inside this gaussian surface? Well, with a charge λ pr unit length and a length L, the charge must be $Q = \lambda L$.
Gauss' law then yields:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{A} = E(r) \cdot 2\pi r L = \frac{Q}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

so that the electric field becomes

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}$$

This is exactly what we found in exercise 2 *d*) in øving 2.

Exercise 4

a) **C**. It follows from Gauss' law that the surface then encloses a net negative charge. On the closed surface, the surface element vector $d\boldsymbol{A}$ is defined as positive when it is directed *outwards*. If $\boldsymbol{E}$ is everywhere directed inwards, all contributions $d\phi = \boldsymbol{E} \cdot d\boldsymbol{A}$ to the flux through the closed surface then become negative. The total flux through the surface must equal the enclosed charge (divided by the constant ε_0), so the enclosed charge must also be negative.

b) **A**. The net electric flux out through the surface must, according to Gauss' law, be equal to the net enclosed charge divided by ε_0. In this case, the net enclosed charge is $q - q = 0$.

c) **B**. Here, the net enclosed charge is $-2q + q = -q$.

d) **B**. The potential from a point charge is $V(r) = q/4\pi\varepsilon_0 r$, choosing $V(r \to \infty) = 0$. With $V = 50$ V, we find that

$$r = \frac{q}{4\pi\varepsilon_0 V} = 9 \cdot 10^9 \cdot \frac{10^{-8}}{50} = 1.8 \text{ m}$$

With SI units for all parameters involved, we are guaranteed that the resulting r will also come out in SI units, i.e., in meters.

e) **D**. The electric field equals the negative gradient of the potential, $\boldsymbol{E} = -\nabla V$. Here, we are told that the potential is constant, and equal to 100V. And the gradient of a constant is zero.

f) **C** According to Gauss' law, the net electric flux out through a closed surface is determined by the net charge inside the surface. Here, the net charge inside the three surfaces is equal, $Q_{\text{in}} = \pi R^2 \sigma$, so the net flux out through the three surfaces must also be equal.

g) **D**

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dr}\hat{r} \quad \Rightarrow \quad \text{graph 5}$$

h) **D**

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dx}\hat{x} = -15\frac{\text{V}}{\text{m}}\hat{x}$$

i) **A**

$$E_x = -\frac{dV}{dx} = -2\frac{\text{V}}{\text{m}}$$

j) **D** According to Gauss' law, the total (net) electric flux through a closed surface enclosing a point charge q equals

$$\phi_{\text{tot}} = \frac{q}{\varepsilon_0}$$

Due to the symmetry of the problem, the same amount of this total flux must pass through the surface of the given cube as the flux that passes through any of the "remaining" 7 cubes that are required to complete the cube with q in the centre. (Remember, 8 octants in a 3-dimensional coordinate system!) Each of these 8 cubes have 3 "adjacent" faces where $\boldsymbol{E}$ is parallel with the surface (i.e., $\boldsymbol{E}$ is perpendicular to the surface normal). Hence, there can be no flux passing through these faces. Furthermore, each of the 8 cubes have 3 equivalent "opposite" ("non-adjacent") faces, and the hatched face is one of these. Without doing any calculations, we can conclude that the same amount of flux must pass through each of these "opposite" faces. The "large cube" (i.e., the one with q in the centre) then has $8 \cdot 3 = 24$ such faces, all of them equivalent with respect to the amount of electric flux passing through them. Therefore, the flux through one of them must simply be

$$\phi = \frac{\phi_{\text{tot}}}{24} = \frac{q}{24\varepsilon_0}$$