Institutt for fysikk, NTNU
TFY4155/FY1303 Elektrisitet og magnetisme
Vår 2004

Solution to øving 6

Guidance Monday February 16

Exercise 1

a) Maximum electric field of 3 MV/m is achieved with a surface charge density

$$\sigma_{\rm max} = \varepsilon_0 E_{\rm max} = 8.85 \cdot 10^{-12} \cdot 3 \cdot 10^6 = 2.7 \cdot 10^{-5} {\rm C/m^2}$$

b) A metal sphere with radius R and charge Q has a surface charge density

$$\sigma = \frac{Q}{4\pi R^2}$$

We see that maximum value of σ corresponds to minimum value of R:

$$R_{\rm min} = \sqrt{\frac{Q}{4\pi\sigma_{\rm max}}} = \sqrt{\frac{1}{4\pi \cdot 2.7 \cdot 10^{-5}}} \simeq 54{\rm m}$$

c) On a surface where the atoms are arranged in a regular quadratic lattice with a distance 0.3 nm between nearest neighbours, each surface atom occupies an area $A_0 = \left(0.3 \cdot 10^{-9}\right)^2 = 9 \cdot 10^{-20}{\rm m}^2$. Thus, the number of surface atoms pr m^2 is $(9 \cdot 10^{-20})^{-1} = 1.1 \cdot 10^{19}$

[If the surface atoms were arranged in a regular *triangular* lattice (with the same nearest neighbour distance 0.3 nm = 3 Å), the area pr surface atom would be a regular *hexagon* with sides $\sqrt{3}$ Å. The area of such a hexagon is $9\sqrt{3}/2$ Å$^2 \simeq 7.8 \cdot 10^{-20}$ m^2, which results in about $1.3 \cdot 10^{19}$ atoms pr m^2. In other words, somewhat more closely packed than with a quadratic lattice!]

d) With a charge pr unit surface area $\sigma_{\rm max} = 2.7 \cdot 10^{-5}{\rm C/m^2}$ and $1.1 \cdot 10^{19}$ surface atoms pr m^2, we have an average charge pr surface atom $2.7 \cdot 10^{-5}/1.1 \cdot 10^{19} = 2.45 \cdot 10^{-24}$C, which is $2.45 \cdot 10^{-24}/1.6 \cdot 10^{-19} = 1.5 \cdot 10^{-5}$ of the charge of an electron. This must also represent the fraction of surface atoms that has an extra electron.

Exercise 2

We have $E = 0$ everywhere inside the metal in electrostatic equilibrium. (See the lecture notes.) This means, according to Gauss' law, that a gaussian surface S (closed surface) that encloses the cavity, and lies completely inside the conductor, encloses zero net charge ($q_{\rm in}$):

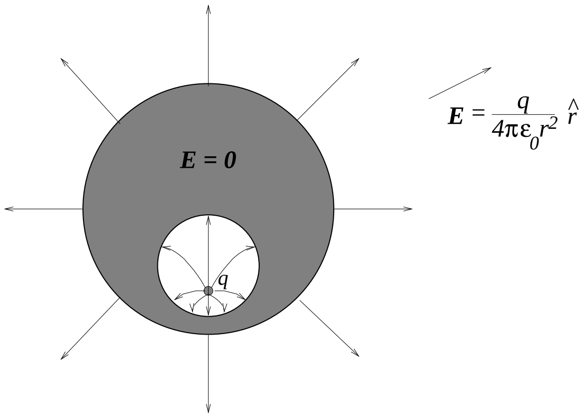
$$E = 0 \quad \Rightarrow \quad q_{\rm in} = \varepsilon_0 \oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = 0$$

This gaussian surface may be located arbitrarily close to the surface of the cavity, so we may conclude that a charge $-q$ has been induced on the surface of the cavity. (Then, the total charge inside the gaussian surface is $q - q = 0$.)

This induced charge must distribute itself over the cavity surface in such a way that the electric field disappears everywhere inside the conductor. In other words, the contribution to the field inside the conductor from the point charge q must precisely be cancelled by the contribution from the induced charge $-q$. Then, I hope, it should be more or less obvious that we must have the largest amount of induced charge at the bottom, where the point charge lies close to the surface, and the smallest amount of induced charge at the top, where the point charge is farther away.
Since the conductor has zero net charge, we must also have an induced charge q on the outer surface of the conductor. (Remember: No net charge inside a conductor in equilibrium!) This charge will distribute itself uniformly over the outer surface because the "asymmetry" caused by the point charge within the cavity is cancelled by the induced charge $-q$ on the cavity surface.
Thus, outside the sphere, we simply "see" a spherically symmetric surface charge, so that the electric field outside the sphere becomes

$$\boldsymbol{E}(r) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

where r is the distance from the center of the sphere. Electric field lines will be roughly like this:



We have chosen to draw 8 field lines pr charge q. All 8 field lines starting on the point charge must terminate on the surface of the cavity, and in such a way that they are perpendicular to the surface. (Electric field always normal to a metal surface!) Inside the conductor, $E = 0$, so here we have no field lines. On the outer surface, we have a charge q uniformly distributed, so here we have again 8 field lines, directed radially outwards.
Comment: It followed directly from Gauss' law and the fact that $E = 0$ inside the conductor, that the total induced charge is $-q$ and q, respectively, on the inner and outer surface of the conductor. However, I guess we haven't really argued sufficiently for how these induced charges distribute themselves. One thing is certain: *Together*, all the charges must distribute themselves so that we obtain $E = 0$ everywhere inside the conductor. Above, I have simply *claimed* that the point charge q and the inner induced charge $-q$ do this job *alone*, without any "help" from the outer induced charge q. Can we be sure that this is at all possible? The answer is yes: Imagine a VERY VERY large metal sphere with a tiny cavity somewhere deep inside, and with a point charge located inside the cavity, as above. Now, all the outer induced surface charge

is simply so far away from the cavity that in order to obtain $E = 0$ inside the conductor (at least in the vicinity of the cavity), the inner induced charge $-q$ must cancel the field from the point charge alone. In other words, it is *possible* to obtain $E = 0$ inside the conductor without any help from the charge on the outer surface. But then we may conclude that this is the *only* possibility, no matter how big or small the metal sphere is. Why is this the only possibility? Well, we have socalled *uniqueness theorems* in electrostatics which guarantee that a *possible* charge distribution is also the *only* possible one. (This is not "pensum"! However, see e.g. Griffiths, chapter 3, if you are interested.)

Exercise 3

a) On the positive charge, there acts a force $\boldsymbol{F}_+ = q\boldsymbol{E}$ and on the negative charge acts a force $\boldsymbol{F}_- = -q\boldsymbol{E}$. The total force must be the sum of these two:

$$\boldsymbol{F} = \boldsymbol{F}_+ + \boldsymbol{F}_- = q\boldsymbol{E} - q\boldsymbol{E} = 0$$

b)

$$\boldsymbol{\tau} = \boldsymbol{r}_+ \times \boldsymbol{F}_+ + \boldsymbol{r}_- \times \boldsymbol{F}_- = \frac{\boldsymbol{a}}{2} \times (q\boldsymbol{E}) + \left(-\frac{\boldsymbol{a}}{2}\right) \times (-q\boldsymbol{E}) = q\boldsymbol{a} \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E}$$

We have chosen positive direction for the angle θ as shown in the figure in the text. Thus, the vector $\boldsymbol{p} \times \boldsymbol{E}$ will, for positive θ between 0 and 180 degrees, point into the paper plane, i.e., in the negative z direction. Thus, we must include a negative sign:

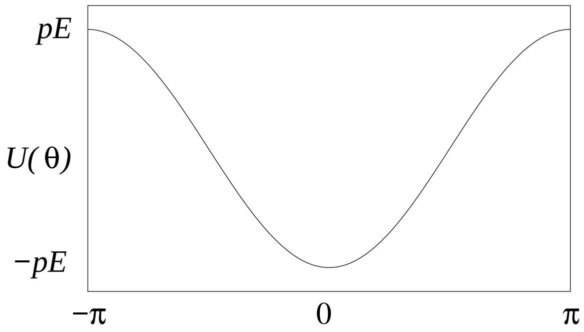
$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E} = -pE \sin\theta \ \hat{z}$$

c) With $\tau = -\partial U/\partial\alpha$, we have $dU = -\tau \ d\alpha$. This means that a small rotation of the dipole through an angle $d\alpha$, under the influence of the torque τ, results in a change dU in potential energy given by $-\tau \ d\alpha$. Thus, the potential energy for a given value of the angle θ between $\boldsymbol{E}$ and $\boldsymbol{p}$, *relative to* a chosen reference $U(\theta_0)$, becomes

$$\begin{aligned} U(\theta) &= \int_{\theta_0}^{\theta} dU \\ &= -\int_{\theta_0}^{\theta} \tau(\alpha) \ d\alpha \\ &= pE \int_{\theta_0}^{\theta} \sin\alpha \ d\alpha \\ &= pE\left(\cos\theta_0 - \cos\theta\right) \\ &= -pE\cos\theta \end{aligned}$$

Here, I chose $U(0) = -pE$, i.e., $\theta_0 = \pi/2$.

Sketch:



We have minimum value for U, and hence a stable equilibrium situation, for $\theta = 0$, i.e., when the dipole is oriented with $\boldsymbol{p}$ along $\boldsymbol{E}$.
In conclusion: Electric dipoles, for example polar molecules in a dielectric, align along an external electric field.