Institutt for fysikk, NTNU
TFY4155/FY1303 Elektrisitet og magnetisme
Vår 2004

Solution to øving 9

Guidance Monday March 8

Exercise 1

a) With spherically symmetric charge distribution, the electric field must be radially directed and only dependent on the distance r from the center of the spheres. With a dielectric present, we first determine the electric displacement $D(r)$ in the region $a < r < b$ with Gauss' law. We use a spherical gaussian surface with radius r. Inside this surface, we have a free charge q, so we find

$$D(r) \cdot 4\pi r^2 = q \quad \Rightarrow \quad D(r) = \frac{q}{4\pi r^2}$$

The relation betwwen D and E is

$$D(r) = \varepsilon E(r)$$

Now, the potential difference between the two conductors can be determined:

$$\begin{aligned}
\Delta V &= V_a - V_b \\
&= -\int_a^b E(r)\, dr \\
&= -\int_a^b \frac{D(r)}{\varepsilon} dr \\
&= \frac{q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) \\
&= \frac{q(b-a)}{4\pi\varepsilon ab}
\end{aligned}$$

The capacitance of this spherical conductor is therefore

$$C = \frac{q}{\Delta V} = \frac{4\pi\varepsilon ab}{(b-a)}$$

If the dielectric layer is very thin, we have $a \simeq b$, and approximately the same area $A \simeq 4\pi ab \simeq 4\pi a^2 \simeq 4\pi b^2$ on the two conductors. Thus, with $d = b - a =$ the distance between the conductors, we have

$$C \simeq \varepsilon \frac{A}{d}$$

i.e., as for a parallel plate capacitor.

b) If we consider a single conducting sphere as the limit $b \to \infty$ in the previous question, and in addition let $\varepsilon \to \varepsilon_0$, we have

$$C = 4\pi\varepsilon_0 a$$

for the capacitance of a sphere with radius a.

Exercise 2

In this exercise, we have a *fixed* potential difference of 100 V between the two metal plates. This means e.g. that the total free charge on a given plate is not the same in the three given cases. See also questions 7 and 12 – 19 in øving 8, where similar things are being discussed.

One of the targets in this exercise is to find the three capacitances C_a, C_b and C_c. With the knowledge from questions 12, 13, 18 and 19 in øving 8, we can perhaps start by writing down what these capacitances must be.
In (a), we have two parallel plate capacitors coupled in parallel, both with plate separation d and plate area $A/2$, one of them filled with air, and the other filled with a dielectric with permittivity ε_1. We may write down the capacitance of each of these two directly:

$$C_a^h = \varepsilon_0 \frac{A}{2d}$$

for the half on the right side, and

$$C_a^v = \varepsilon_1 \frac{A}{2d}$$

for the half on the left side. In question 12, øving 8, we found that the total capacitance of capacitors coupled in parallel is computed by adding the partial capacitances. Thus, we have

$$C_a = C_a^v + C_a^h = \varepsilon_0 \frac{A}{2d} (\varepsilon_{r1} + 1)$$

In (b), we have two parallel plate capacitors coupled in series, both with plate separation $d/2$ and plate area A, one of them filled with a dielectric with permittivity ε_2 and the other filled with a dielectric with permittivity ε_3. The capacitance of each of these may be written down directly:

$$C_b^o = \varepsilon_2 \frac{2A}{d}$$

for the upper half and

$$C_b^n = \varepsilon_3 \frac{2A}{d}$$

for the lower half. In question 13, øving 8, we found that the total inverse capacitance of capacitors coupled in series is computed by adding inverse partial capacitances. Thus, we have

$$C_b = \left(\frac{1}{C_b^o} + \frac{1}{C_b^n} \right)^{-1} = \varepsilon_0 \frac{2A}{d} \cdot \frac{\varepsilon_{r2}\varepsilon_{r3}}{\varepsilon_{r2} + \varepsilon_{r3}}$$

In (c), we have two parallel plate capacitors coupled in parallel. One of them has plate separation d, plate area $A/2$ and is filled with a dielectric with permittivity ε_1. The other consists of two capacitors coupled in series, as in (b), but with plate area $A/2$ instead of A. The total capacitance must therefore be

$$\begin{aligned} C_c &= \varepsilon_0 \varepsilon_{r1} \frac{A}{2d} + \varepsilon_0 \frac{A}{d} \cdot \frac{\varepsilon_{r2}\varepsilon_{r3}}{\varepsilon_{r2} + \varepsilon_{r3}} \\ &= \varepsilon_0 \frac{A}{d} \left(\frac{\varepsilon_{r1}}{2} + \frac{\varepsilon_{r2}\varepsilon_{r3}}{\varepsilon_{r2} + \varepsilon_{r3}} \right) \end{aligned}$$

With the given numerical values, we have $A = 10^{-3}$ m^2 and $d = 10^{-3}$ m, so that $A/d = 1$ m. Further, we have $\varepsilon_0 = 8.85 \cdot 10^{-12}$ F/m $= 8.85$ pF/m. The three capacitances thus become

$$\begin{aligned}
C_a &= \varepsilon_0 \cdot \frac{1}{2} \cdot (4+1) = 2.5\varepsilon_0 = 22.1 \\
C_b &= \varepsilon_0 \cdot 2 \cdot \frac{6 \cdot 2}{6+2} = 3\varepsilon_0 = 26.6 \\
C_c &= \varepsilon_0 \cdot 1 \cdot \left(\frac{4}{2} + \frac{6 \cdot 2}{6+2}\right) = 3.5\varepsilon_0 = 31.0
\end{aligned}$$

all in units of pF.

Now, back to the physics! Let's first look qualitatively at what happens.
In (a), the dielectric on the left side is polarized in the electric field from the two metal plates. Then we know that the net result becomes an induced surface charge on each side of the dielectric, in this case negative on top and positive at the bottom, since the upper and lower metal plate has positive and negative (free) charge, respectively. In comparison with a capacitor completely filled with air, with uniformly distributed free charge over the area of the metal plates, we have now disturbed the original equilibrium, but this equilibrium is restored by the transfer of some free charge from the right half of the metal plates to the left half. In equilibrium, a given metal plate must have constant potential all over. Hence, the electric field E_a must also be the same on the right and on the left side of the capacitor. Concerning the charge distribution, this means that the *total* charge on the right and the left side must be the same. Let us call the density of free charge σ^v_{af} and σ^h_{af} on the left and on the right side, respectively, and the density of induced (bound) charge σ^v_{ai} on the left side. (On the right side, we have no polarization, and therefore also no induced charge.) We also use a notation where all the different σ are *positive*. Thus, we must have

$$\sigma^h_{af} = \sigma^v_{af} - \sigma^v_{ai}$$

in order to have the same amount of total charge on the right and on the left side.
In (b), all the volume between the plates is polarized, and we end up with induced surface charge on both sides of both dielectrics. The upper dielectric has the largest value of the relative permittivity. This means we get a stronger polarization in this region than in the lower half. Hence, the induced charge on the surface of the upper dielectric, σ^o_{bi}, will be larger than the induced charge on the surface of the lower dielectric, σ^n_{bi}. The sign of the various induced charges must be negative next to the upper metal plate, positive next to the lower metal plate, and therefore positive on the lower surface of the upper dielectric, and negative on the upper surface of the lower dielectric (since they are both overall electrically neutral). In this case, there is no difference between right and left, so the free charge remains uniformly distributed over the metal plates. Furthermore, the line integral of the electric field along a path between the two plates must still have the value $\Delta V = 100$ V. With air between the plates, we would have had a uniform electric field $E_0 = \Delta V/d = 100/10^{-3} = 10^5$ V/m $= 100$ kV/m $= 100$ V/mm. In the present situation, we have concluded that we have the strongest polarization in the upper half. Therefore, the electric field, E^o_b, must here be weaker than the field E^n_b in the lower half. It should be claer that E^n_b must be larger than 100 V/mm and also that E^o_b must

be smaller than 100 V/mm, and all together in such a way that

$$E_b^o \cdot \frac{d}{2} + E_b^n \cdot \frac{d}{2} = 100 \text{ V/mm}$$

Also in this case, if we compare with the situation without any dielectric present, additional charge must have arrived at the metal plates in order to sustain the same potential difference between the plates. It is the voltage source that supplies this additional charge.
Capacitor c is essentially a combination of a and b, and we have induced surface charge on all the three dielectric media, as discussed for (a) and (b) above: σ_{ci}^v in the left half, σ_{ci}^{oh} on the upper dielectric in the right half, and σ_{ci}^{nh} on the lower dielectric in the right half. These induced charges must of course be of equal magnitude as the corresponding ones in (a) and (b):

$$\begin{aligned} \sigma_{ci}^v &= \sigma_{ai}^v \\ \sigma_{ci}^{oh} &= \sigma_{bi}^o \\ \sigma_{ci}^{nh} &= \sigma_{bi}^n \end{aligned}$$

Furthermore, the electric field in the left half must be the same as in the left half in (a),

$$E_c^v = E_a^v$$

and the electric field in the upper and lower half on the right side must be the same as the corresponding fields in (b),

$$\begin{aligned} E_c^{oh} &= E_b^o \\ E_c^{nh} &= E_b^n \end{aligned}$$

Now, we have essentially a complete overview of the situation! We have also established a notation and various relations between the different quantities. In addition to all this, we know (see lecture notes) that the electric displacement D equals the density of free charge, and that the electric polarization P equals the density of induced charge. Finally, we have the relations $D = \varepsilon_0 E + P = \varepsilon E$. For our systems, we could here safely drop all the vector signs: All vectors $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{P}$ point *downwards*. Now, we have enough information to compute all the interesting quantities for the three capacitors.

(a) Constant electric field E_a between the plates:

$$\Delta V = E_a \cdot d \;\Rightarrow\; E_a = \frac{\Delta V}{d} = 100 \text{ V/mm}$$

Electric displacement:

$$D_a^h = \varepsilon_0 E_a = 8.85 \cdot 10^{-12} \cdot 10^5 = 885 \cdot 10^{-9} \text{ C/m}^2 = 885 \text{ nC/m}^2$$

in the right half, and

$$D_a^v = \varepsilon_1 E_a = 4 \cdot 8.85 \cdot 10^{-12} \cdot 10^5 = 3540 \cdot 10^{-9} \text{ C/m}^2 = 3540 \text{ nC/m}^2$$

in the left half. This is also the free charge pr unit area on the right and on the left half of the metal plates:

$$\begin{aligned}
\sigma^h_{af} &= D^h_a = 885 \text{ nC/m}^2 \\
\sigma^v_{af} &= D^v_a = 3540 \text{ nC/m}^2
\end{aligned}$$

The difference between these two must be equal to the induced charge pr unit area on the left side, which is the same as the polarization P^v_a on the left side:

$$\sigma^v_{ai} = P^v_a = \sigma^v_{af} - \sigma^h_{af} = 2655 \text{ nC/m}^2$$

(b) Here, we have constant electric displacement between the plates:

$$D^o_b = D^n_b = D_b = \sigma^o_{bf} = \sigma^n_{bf} = \sigma_{bf}$$

We cannot say immediately what the free charge σ_{bf} on the metal plates is, but we know the relation between D and E in the two layers, and also the relation between E and ΔV. All together:

$$\begin{aligned}
\Delta V &= \frac{d}{2}\left(E^o_b + E^n_b\right) \\
&= \frac{d}{2}\left(\frac{D^o_b}{\varepsilon_2} + \frac{D^n_b}{\varepsilon_3}\right) \\
&= \frac{D_b d}{2\varepsilon_0}\left(\frac{1}{\varepsilon_{r2}} + \frac{1}{\varepsilon_{r3}}\right) \\
\Rightarrow D_b &= \frac{2\varepsilon_0 \Delta V}{d}\left(\frac{1}{\varepsilon_{r2}} + \frac{1}{\varepsilon_{r3}}\right)^{-1} \\
&= 2 \cdot 8.85 \cdot 10^{-12} \cdot 10^5 \left(\frac{1}{6} + \frac{1}{2}\right)^{-1} \\
&= 2655 \text{ nC/m}^2
\end{aligned}$$

Electric field in the upper half:

$$E^o_b = \frac{D_b}{\varepsilon_2} = \frac{2655 \cdot 10^{-9}}{6 \cdot 8.85 \cdot 10^{-12}} = 50 \text{ kV/m } = 50 \text{ V/mm}$$

Electric field in the lower half:

$$E^n_b = \frac{D_b}{\varepsilon_3} = \frac{2655 \cdot 10^{-9}}{2 \cdot 8.85 \cdot 10^{-12}} = 150 \text{ kV/m } = 150 \text{ V/mm}$$

Polarization, and therefore also induced charge, "upstairs":

$$P^o_b = D_b - \varepsilon_0 E^o_b = D_b - \frac{D_b}{\varepsilon_{r2}} = \frac{5}{6}D_b = 2212.5 \text{ nC/m}^2 = \sigma^o_{bi}$$

Polarization, and therefore also induced charge, "downstairs":

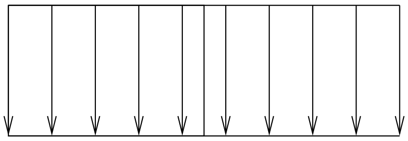
$$P^n_b = D_b - \varepsilon_0 E^n_b = D_b - \frac{D_b}{\varepsilon_{r3}} = \frac{1}{2}D_b = 1327.5 \text{ nC/m}^2 = \sigma^n_{bi}$$

(c) Here, the various quantities on the left side will be as in the left part of (a), and in the right half, the same as in (b).

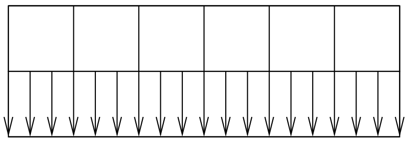
Field lines:

Feltlinjer for $\boldsymbol{E}$:

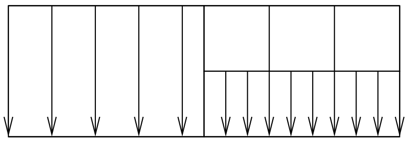
(a)



(b)

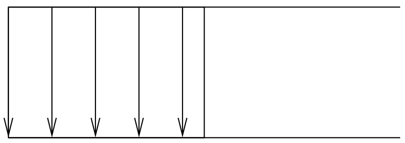


(c)

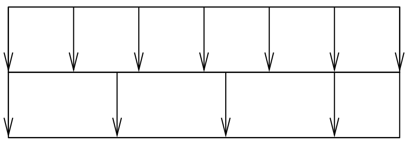


Feltlinjer for $\boldsymbol{P}$:

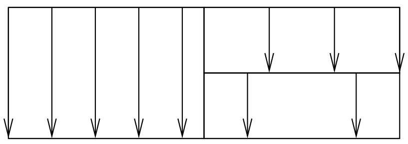
(a)

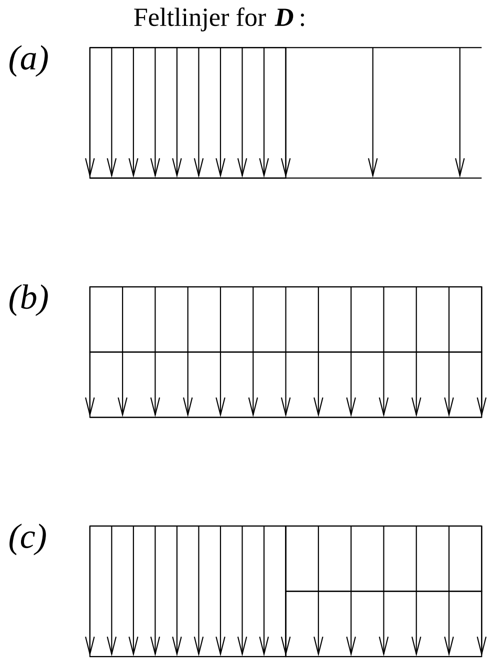


(b)

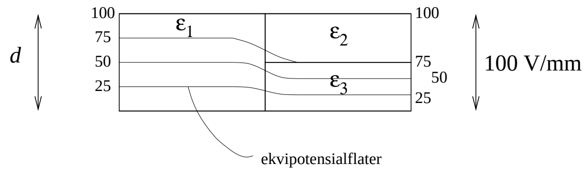


(c)





The extra question:
So, we have found that the electric field is 100V/mm in the left half, 50 V/mm in the upper part of the right half, and 150 V/mm in the lower part of the right half in capacitor (c). With $V = 100$V on the upper plate and $V = 0$ V on the lower plate, this means that half way between the two metal plates, the potential has the value 50 V on the left side, but the value 75 V on the right side. This is OK as long as we are sufficiently far away from the vertical midplane. But: The potential must be continuous everywhere! For example, the equipotential surface with value 75 V, which lies 0.25 mm below the upper plate on the left side, cannot suddenly jump down to a position 0.50 mm below the upper plate when we cross the vertical midplane. We must have some kind of "smooth" transition when we cross the midplane, roughly as shown in the figure:



From the shape of the sketched equipotential surfaces, we see that the electric field must have a horizontal component in the vicinity of the vertical midplane, since $\boldsymbol{E} = -\nabla V$ is always perpendicular to the equipotential surfaces.