**Institutt for fysikk, NTNU**
**TFY4155/FY1003 Elektrisitet og magnetisme**
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# Solution to øving 12

Guidance Thursday April 7 and Friday April 8

*Exercise 1*

In the first experiment, $B = 0$. Then, Newton's second law is:

$$\begin{aligned}
\boldsymbol{F} &= m\boldsymbol{a} = q\boldsymbol{E} \\
\Rightarrow \frac{d\boldsymbol{v}}{dt} &= \frac{q}{m}\boldsymbol{E} \\
\Rightarrow \boldsymbol{v}(t) &= \boldsymbol{v}(0) + \frac{q}{m}\boldsymbol{E}t = \frac{d\boldsymbol{r}}{dt} \\
\Rightarrow \boldsymbol{r}(t) &= \boldsymbol{r}(0) + \boldsymbol{v}(0)t + \frac{q}{2m}\boldsymbol{E}t^2
\end{aligned}$$

Here, it is natural to choose $t = 0$ the moment the particle enters the region with $E \neq 0$, and furthermore, to choose the origin in this position:

$$\boldsymbol{r}(0) = (x_0, y_0) = (0, 0)$$

Here, the velocity is

$$\boldsymbol{v}(0) = v\,\hat{x}$$

when we orient the $x$ axis towards the right. The $y$ axis is oriented upwards, so that

$$\boldsymbol{E} = -E\,\hat{y}$$

(i.e., with $E > 0$) The particle trajectory thus becomes a parabola, just like when we throw a mass in the field of gravity. The velocity in the $x$ direction is not affected by the electric field, so

$$x(t) = vt$$

whereas the particle obtains a constant acceleration in the $y$ direction, i.e., the displacement in the $y$ direction, as a function of $t$, must be determined by

$$y(t) = -\frac{q}{2m}Et^2$$

The particle will leave the region where $E \neq 0$ at the moment

$$t_L = \frac{x(t_L)}{v} = \frac{L}{v}$$

The vertical position is then

$$y(t_L) = -\frac{q}{2m}E\frac{L^2}{v^2}$$

Already, we may conclude that $q < 0$ if $y(t_L) > 0$.
The distance from $x = L$ to $x = L + D$ is then traveled without influence from any kind of forces, and with a direction relative to the $x$ axis in terms of the angle $\alpha$, where

$$\tan\alpha = \frac{v_y(t_L)}{v_x(t_L)} = \frac{-\frac{q}{m}E\frac{L}{v}}{v} = -\frac{qEL}{mv^2}$$

Besides, we must have

$$\tan\alpha = \frac{y - y(t_L)}{D}$$

where $y$ is where the electron hits the detector, at $x = L + D$.

The experiment is then repeated with the same $E$-field, but now we turn on a magnetic field $B$ directed into the plane, so that the particles are no longer deflected by the fields. This implies that the electric force (upwards) is exactly balanced by the magnetic force (downwards). In other words:

$$\begin{aligned} \boldsymbol{F} &= q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B} = 0 \\ \Rightarrow E &= vB \\ \Rightarrow \frac{1}{v} &= \frac{B}{E} \end{aligned}$$

Hence:

$$\begin{aligned} \frac{y - y(t_L)}{D} &= -\frac{qEL}{mv^2} = -\frac{qEL}{m} \cdot \frac{B^2}{E^2} \\ \Rightarrow y + \frac{q}{2m}EL^2\frac{B^2}{E^2} &= -\frac{qEL}{m} \cdot \frac{B^2}{E^2}D \\ \Rightarrow yE &= -\frac{q}{m} \cdot B^2\left(DL + \frac{1}{2}L^2\right) \\ \Rightarrow \frac{q}{m} &= -\frac{yE}{B^2\left(DL + \frac{1}{2}L^2\right)} \end{aligned}$$

I.e.,

$$a = \frac{E}{B^2\left(DL + L^2/2\right)}$$

*Exercise 2*

*a*) The speed of the ions when they enter the region with magnetic field is determined by the change in potential energy, going through the voltage difference $V$, being equal to the change in the kinetic energy of the ions:

$$eV = \frac{1}{2}mv^2 \quad \Rightarrow \quad v = \sqrt{\frac{2eV}{m}}$$

The centripetal acceleration inside the magnetic field is

$$a = \frac{v^2}{r}$$

so that Newton's 2. law gives

$$F = m\frac{v^2}{r} = evB \quad \Rightarrow \quad r = \frac{mv}{eB}$$

Radius for the resulting circular path for a particle with mass $m$ becomes

$$r = \frac{1}{B}\sqrt{\frac{2Vm}{e}}$$

i.e., proportional with $\sqrt{m}$. Radii and masses for the different isotopes must be related as follows:

$$\frac{r_i}{r_j} = \sqrt{\frac{m_i}{m_j}}$$

where $i, j = 79$ or 81.
If the points where the ions hit the photographic plate are supposed to be separated by a distance of (at least) $a = 1.0$ cm, the *diameter* of the two circular paths must differ by 1.0 cm. We obtain

$$a = 1.0 \text{ cm } = 2(r_{81} - r_{79}) = 2r_{79}\left(\sqrt{\frac{m_{81}}{m_{79}}} - 1\right)$$

This gives

$$r_{79} = \frac{a}{2}\left(\sqrt{\frac{m_{81}}{m_{79}}} - 1\right)^{-1} = 0.5 \text{ cm } \cdot \left(\sqrt{\frac{81}{79}} - 1\right)^{-1} \simeq 39.7 \text{ cm}$$

and

$$r_{81} = r_{79} + \frac{a}{2} \simeq 40.2 \text{ cm}$$

Now, we can determine how strong magnetic field that can be used to achieve these radii:

$$B = \frac{1}{r_{81}}\sqrt{\frac{2Vm_{81}}{e}} = \frac{1}{0.402} \cdot \sqrt{\frac{2 \cdot 400 \cdot 81 \cdot 1.67 \cdot 10^{-27}}{1.6 \cdot 10^{-19}}} = 0.065 \text{ T}$$

This represents the upper limit of $B$: A stronger magnetic field will reduce both $r_{79}$ and $r_{81}$, but $r_{81}$ the most, so that the "hit points" move closer to each other. However, at the same time the diameter $d_{81} = 2r_{81}$ must not be larger than the physical limit of the instrument, given by $L = 250cm$. That corresponds to a minimum value of the magnetic field strength:

$$B_{\min} = \frac{1}{L/2}\sqrt{\frac{2Vm_{81}}{e}} = \frac{1}{1.25} \cdot \sqrt{\frac{2 \cdot 400 \cdot 81 \cdot 1.67 \cdot 10^{-27}}{1.6 \cdot 10^{-19}}} = 0.021 \text{ T}$$

In other words, we may use a magnetic field between 21 and 65 mT.

*Exercise 3*

*a*) In this exercise, we may argue somewhat along the same lines as we did when we calculated the electric field on the symmetry axis of a uniformly charged ring. In that case, we looked at the contributions to the total field from diametrically opposite charge elements $dq$ and convinced ourselves that the total electric field had to be directed along the symmetry axis.

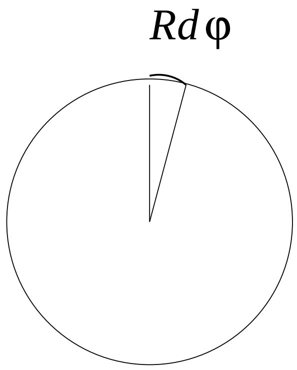
Here, we may e.g. look at the two current elements lying exactly on the positive and the negative $y$ axis, respectively, and determine the direction of the contribution to the magnetic field on the $z$ axis from these two. Let us look at positive values of $z$ first. (See figure below.) The "current element" $I\ d\boldsymbol{l}$ which crosses the positive $y$ axis has its direction along the negative $x$ axis. The cross product of this vector with $\boldsymbol{r}$ from the current element to the actual position on the positive $z$ axis becomes a vector lying in the $yz$ plane, with positive $y$ and $z$ components. The diametrically opposite current element, i.e., the one crossing the negative $y$ axis, is directed along the positive $x$ axis. The cross product of this vector with $\boldsymbol{r}$ from the current element to the actual position on the positive $z$ axis becomes a vector which also lies in the $yz$ plane, but this vector will have a *negative* $y$ component and positive $z$ component. From symmetry reasons, these two contributions to $\boldsymbol{B}$ must be equal in absolute value, have equal $z$ components (with the same sign), and have equal $y$ components, but with the opposite sign. Thus, the sum of these two contributions must point along the positive $z$ axis.
In this way, we could argue for any pair of diametrically opposite current elements around the ring. They will all have an equal $z$ component with the same sign, and equal $x$ and $y$ components with opposite sign.
Conclusion: $\boldsymbol{B}$ on the positive $z$ axis is directed along the positive $z$ axis.

*b*) In *a*), we convinced ourselves that $\boldsymbol{B}(z)$ is directed along the positive $z$ axis if $z > 0$. What about $z < 0$?
Looking again at the figure below, we see that the current element that crosses the positive $y$ axis will give a contribution to $\boldsymbol{B}(z)$ on the negative $z$ axis which lies in the $yz$ plane, with positive $z$ component and negative $y$ component. For the current element crossing the negative $y$ axis, we find a contribution with positive $z$ component and positive $y$ component. Altogether, a magnetic field directed along the positive $z$ axis.
Conclusion: The magnetic field points along the postive $z$ axis everywhere on the $z$ axis.

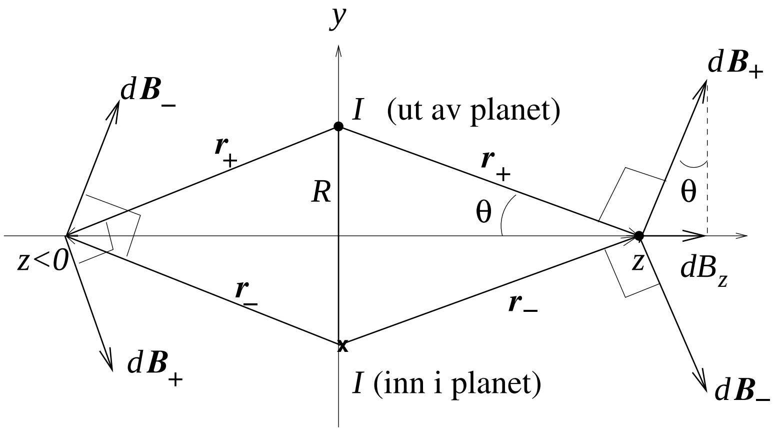
*c*) The vectors $I\ d\boldsymbol{l}$ and $\hat{r}$ are perpendicular to each other. Thus

$$|I\ d\boldsymbol{l} \times \hat{r}| = IR\ d\phi \cdot 1$$

since a line element $dl$ along a circle equals the radius $R$ multiplied with the "angle element" $d\phi$:




The direction of $d\boldsymbol{B}$ must be as shown in the figure:




From this figure we notice that

$$\frac{dB_z}{dB} = \sin\theta = \frac{R}{r} = \frac{R}{\sqrt{z^2 + R^2}}$$

and this is precisely the $z$ component of the magnetic field that we are looking for. The absolute value of $d\boldsymbol{B}$ becomes

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{IR\,d\phi}{z^2 + R^2}$$

so that

$$dB_z = dB\sin\theta = \frac{\mu_0}{4\pi} \cdot \frac{IR\,d\phi}{z^2 + R^2} \cdot \frac{R}{\sqrt{z^2 + R^2}} = \frac{\mu_0 IR^2\,d\phi}{4\pi\,(z^2 + R^2)^{3/2}}$$

The total $z$ component, and therefore the total magnetic field, is obtained by integrating the contributions from all the current elements around the ring, i.e., by integrating this expression over the angle $\phi$ from 0 to $2\pi$:

$$B(z) = \int dB_z = \frac{\mu_0 IR^2}{4\pi\,(z^2 + R^2)^{3/2}} \int_0^{2\pi} d\phi = \frac{\mu_0 IR^2}{2\,(z^2 + R^2)^{3/2}}$$

which is what we were supposed to show.

$d$) Far away form the current loop, we may write

$$z^2 + R^2 \simeq z^2$$

Thus, the magnetic field is approximately

$$B(z) \simeq \frac{\mu_0 IR^2}{2z^3}$$

The magnetic dipole moment of the current loop is

$$m = IA = I \cdot \pi R^2$$

so we may write this magnetic field as

$$B(z) = \frac{\mu_0 m}{2\pi z^3}$$

It is well worth comparing this result with the electric field on the axis of an electric dipole, in large distance $z$ from the dipole. This is something we already did in øving 5, where we found

$$E(z) = \frac{p}{2\pi\varepsilon_0 z^3}$$

Here, $p$ is the electric dipole moment of the dipole. In other words, exactly the same result, with $m$ instead of $p$, and $\mu_0$ instead of $1/\varepsilon_0$.
We will find more analogies between electrostatics and magnetostatics as we move along!