Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Vår 2005

Solution to øving 13

Guidance Thursday April 14 and Friday April 15

Exercise 1

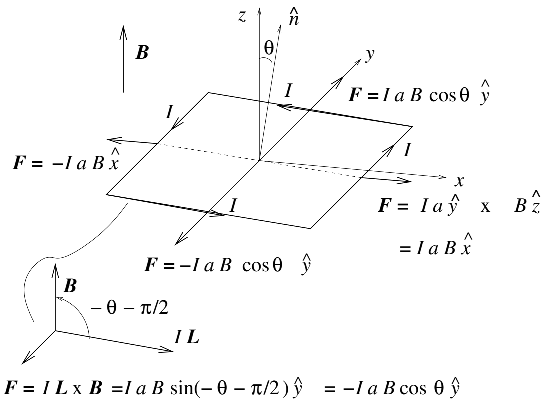
a) The magnetic dipole moment of the loop:

$$\boldsymbol{m} = IA\,\hat{n} = Ia^2\,\hat{n}$$

The loop consists of 4 straight wires of length a, pairwise with the current in the same direction. Hence, the magnetic force,

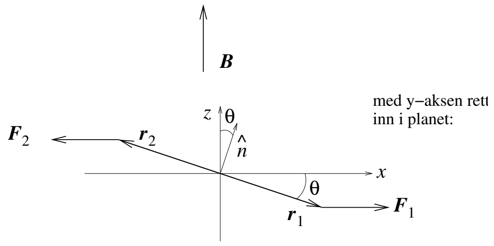
$$\boldsymbol{F} = I\int d\boldsymbol{l}\times\boldsymbol{B} = I\boldsymbol{L}\times\boldsymbol{B}$$

has opposite direction, but is equal in magnitude, for such pairs of straight wires ("current carrying conductor elements"). Thus, the total force on the current loop is zero. Some details are included in this figure:



We see from the figure that the forces from the magnetic field would have deformed the current loop if that had been a possibility. For a macroscopic current loop, this is usually a negligible effect, but if the current loop is a classical model of an electron in an orbit around a nucleus, we see that in addition to an alignment of the current loop (which is the topic of this exercise), the magnetic field will influence the orbital motion of the electron around the nucleus. In other words: The magnetic moment changes both in direction and in absolute value. The first effect is *paramagnetism*, the second effect is *diamagnetism*. A classical model of diamagnetism is the topic in exercise 3, here we concentrate on the orientation of $\boldsymbol{m}$.

b) From the figure above, we see that the two currents running parallel to the xz-plane are influenced by forces in positive and negative y direction, respectively. These forces will then not contribute to the torque around the y axis.
The forces acting on the currents running parallel to the y axis results all together in a torque (see figure below and above)



$$\begin{aligned}
\boldsymbol{\tau} &= \sum \boldsymbol{r} \times \boldsymbol{F} \\
&= -r_1 F_1 \sin\theta \; \hat{y} - r_2 F_2 \sin\theta \; \hat{y} \\
&= -2 \cdot \frac{a}{2} \cdot IaB \sin\theta \; \hat{y} \\
&= -Ia^2 \cdot B \sin\theta \; \hat{y} \\
&= -m \cdot B \sin\theta \; \hat{y} \\
&= \boldsymbol{m} \times \boldsymbol{B}
\end{aligned}$$

The change of sign in the last line is because we had chosen a positive angle θ between the z axis and $\hat{n}$, i.e., between $\boldsymbol{B}$ and $\boldsymbol{m}$. The cross product $\boldsymbol{m} \times \boldsymbol{B}$ is, by definition, m times B times the sine of the angle between $\boldsymbol{m}$ and $\boldsymbol{B}$, i.e., $mB\sin(-\theta) = -mB\sin\theta$.

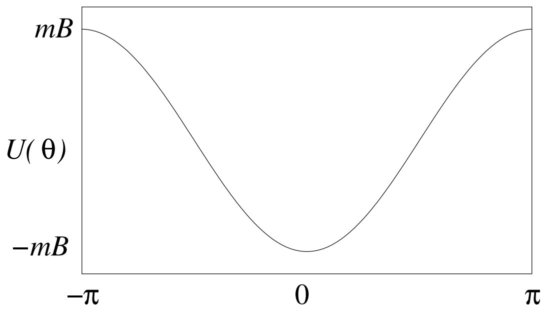
c) In exercise 3c in øving 6 we derived a *general* relation between the torque τ and the corresponding potential energy U, namely that a rotation through an angle $d\alpha$ under the influence of a torque τ results in a change dU in potential energy given by

$$dU = -\tau \; d\alpha$$

The derivation of this relation did not depend upon what *kind* of forces and torques we are talking about, and must therefore be valid also for our magnetic dipole in a magnetic field. Thus:

$$\begin{aligned}
U(\theta) &= \int_{\theta_0}^{\theta} dU \\
&= -\int_{\theta_0}^{\theta} \tau(\alpha) \; d\alpha \\
&= mB \int_{\theta_0}^{\theta} \sin\alpha \; d\alpha \\
&= mB \left(\cos\theta_0 - \cos\theta\right) \\
&= -mB\cos\theta \\
&= -\boldsymbol{m} \cdot \boldsymbol{B}
\end{aligned}$$

Here, I chose $U(0) = -mB$, i.e., $\theta_0 = \pi/2$.
Sketch:



We have minimal U, and thus a stable equilibrium for $\theta = 0$, i.e., when the dipole is oriented so that $\boldsymbol{m}$ is parallel with $\boldsymbol{B}$. We have maximum value of U, and therefore an unstable equilibrium for $\theta = \pm\pi$, i.e., when the dipole is oriented so that $\boldsymbol{m}$ is parallel with $-\boldsymbol{B}$.
This is the analogy in magnetism to polarization of dielectric media in an external electric field: Magnetic dipoles align in the external magnetic field. As mentioned above, this is what we call *paramagnetism*. We have talked about different kinds of magnetism in the lectures – here is a brief summary:
Materials that consist of atoms which have an atomic magnetic dipole moment which is *not* zero, and where the dipole moments on atoms close to each other to *not* interact with each other, are *paramagnets*. *Without* an external magnetic field, the atomic magnetic dipole moments will be oriented in random directions, so that the average *magnetization*, i.e., the average magnetic dipole moment pr unit volume (see item *d*)), becomes zero everywhere in the material. We had exactly the same situation concerning average polarization in a dielectric when we had zero external electric field. *With* an external magnetic field, we obtain a tendency of alignment of magnetic dipole moments along the external field, and thereby an average magnetization different from zero. Examples of paramagnetic materials are aluminum (Al) and magnesium (Mg).
Materials consisting of atoms with *zero* atomic magnetic dipole moments have nothing to align in an external magnetic field. However, as mentioned above, the *orbital motion* of the electrons around the nucleus will be affected by an external magnetic field, so that we have an *induced* magnetic dipole moment in each atom. Such materials are *diamagnets*. We will look qualitatively on this effect in exercise 3 below and find that the induced magnetic dipole moment will always be directed *opposite* to the external field. Diamagnetism is a much weaker effect than paramagnetism. Also in paramagnets, where we do have permanent atomic magnetic dipole moments, we have this diamagnetic "response" in an external magnetic field. However, the diamagnetic response will be almost negligible in a paramagnet. In order to be able to measure diamagnetism, we need a material with zero atomic magnetic dipole moments at the outset (i.e., before we switch on the external field). Examples of diamagnets are gold (Au), silver (Ag) and copper (Cu).
In some materials, we have atoms with magnetic dipole moments that *interact* with the dipole moments on the neighbouring atoms. For example, the interaction may be such that it is energetically favored that neighbouring atoms have their magnetic dipole moments in the same direction. Then we have a *ferromagnet*, and examples of ferromagnets are iron (Fe), cobalt (Co) and nickel (Ni).

d) The maximum density of magnetic dipole moment in iron is equal to the number of iron atoms pr unit volume times the magnetic dipole moment pr iron atom:

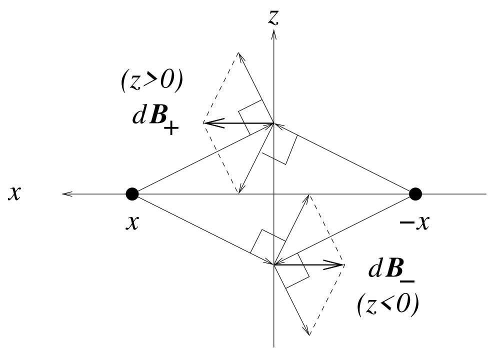
$$\frac{m}{V} = 2\mu_B \cdot \frac{7.9 \cdot 10^6}{55.9} \cdot 6.02 \cdot 10^{23} = 2 \cdot 9.27 \cdot 10^{-24} \cdot \frac{7.9 \cdot 10^6}{55.9} \cdot 6.02 \cdot 10^{23} = 1.6 \cdot 10^6$$

The SI unit of m is Am^2, the unit of V is m^3. Hence, the unit of magnetic dipole moment pr unit volume, or *magnetization*, is A/m.

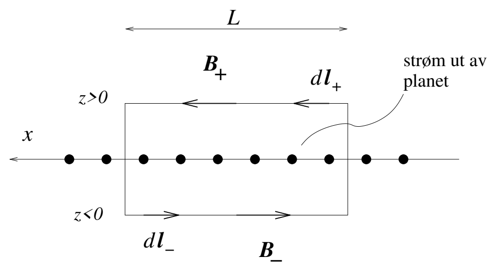
Exercise 2

The direction of $\boldsymbol{B}$:

- $B_y = 0$ because $d\boldsymbol{B} \sim \hat{y} \times \hat{r} \perp \hat{y}$ according to Biot-Savart's law. (All the contributions to the current run in the y direction.)
- $B_z = 0$: Look at the figure below. Here, $d\boldsymbol{B}_+$ and $d\boldsymbol{B}_-$ are contributions to the magnetic field above and below the xy plane, respectively, from "symmetrically located" infinitely long, thin current carrying wires in position $\pm x$. The Biot-Savart law and inspection of the figure then yields the result that $\boldsymbol{B}$ must point in the positive x direction for $z > 0$ and in the negative x direction for $z < 0$.



The absolute value of the magnetic field cannot depend upon x or y when the current carrying plane is infinite. Further, B must have the same absolute value a distance z above the xy plane as a distance z below the same plane. ($B_+ = B_- = B$, see figure below) Then the best choice of amperian loop should be clear: A rectangle with the surface normal in the positive y direction, symmetrically located with respect to the xy plane:



When the integration curve is chosen as shown in the figure, the current enclosed by the amperian loop is *positive*, according to the right hand rule. With a length L in the x direction, the enclosed current is $I_{\text{in}} = i \cdot L$. Then, Ampere's law yields:

$$\oint \boldsymbol{B} \cdot \, d\boldsymbol{l} = 2 \cdot B \cdot L = \mu_0 i \cdot L$$

or

$$B = \mu_0 i/2$$

(On the vertical pieces of the amperian loop, we have $\boldsymbol{B} \perp \, d\boldsymbol{l}$ so these give zero contribution to the integral.)
Alternatively, we could initially have located the complete amperian rectangle on one side of the xy plane. Then we would have had zero enclosed current, and thereby found that B must be independent of z. Next, we locate the amperian loop so that it encloses a part of the xy plane, and therefore a certain current, and finally find the same answer as above. As far as I can see, it is sufficient to use the Ampere law *once* when we choose the curve symmetric with respect to the xy plane.

Exercise 3

a) The centripetal acceleration is v_0^2/R whereas the Coulomb force is $e^2/4\pi\varepsilon_0 R^2$. Then, Newton's 2. law gives

$$m_e \frac{v_0^2}{R} = \frac{e^2}{4\pi\varepsilon_0 R^2} \quad \Rightarrow \quad R = \frac{e^2}{4\pi\varepsilon_0 m_e v_0^2}$$

The orbital angular moment of the electron is

$$\boldsymbol{L}_0 = m_e \, \boldsymbol{r} \times \boldsymbol{v}_0 = m_e R \, v_0 \, \hat{z}$$

while its magnetic dipole moment is

$$\boldsymbol{m}_0 = I\boldsymbol{A} = -\frac{e}{2\pi R/v_0} \cdot \pi R^2 \, \hat{z} = -\frac{1}{2} e v_0 R \, \hat{z} = -\frac{e}{2m_e} \boldsymbol{L}_0$$

b) The chosen direction of $\boldsymbol{B}$ implies that the magnetic force $\boldsymbol{F}_m = -e \, \boldsymbol{v} \times \boldsymbol{B}$ is directed towards the nucleus, i.e., in the same direction as the attractive Coulomb force. Assuming the orbital radius R does not change, the speed v is determined by the equation

$$m_e \frac{v^2}{R} = \frac{e^2}{4\pi\varepsilon_0 R^2} + evB$$

This is a 2. order equation for v,

$$v^2 - \frac{eBR}{m_e} \, v - \frac{e^2}{4\pi\varepsilon_0 m_e R} = 0,$$

with solution

$$v = \frac{eBR}{2m_e} + \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e R} + \left(\frac{eBR}{2m_e}\right)^2}$$

(The solution with a negative sign in front of the square root is negative and not applicable.)
Let us go back and look at the speed without the magnetic field:

$$v_0 = \sqrt{\frac{e^2}{4\pi\varepsilon_0 m_e R}}$$

We see immediately that $v > v_0$. This means that the magnetic dipole moment

$$\boldsymbol{m} = -\frac{1}{2} evR\,\hat{z}$$

is larger than before we turned on the magnetic field. In other words, the *change*

$$\Delta\boldsymbol{m} = \boldsymbol{m} - \boldsymbol{m}_0$$

is directed opposite to the external magnetic field.
If the magnetic field instead was directed downwards, $\boldsymbol{B} = -B\,\hat{z}$, the magnetic force would be directed radially *outwards*, i.e., in the opposite direction of the Coulomb force, so that the additional term evB in the equation of motion would enter with the opposite sign. Thus, the new speed v would have become smaller than the initial speed v_0, and the magnetic dipole moment also smaller than before we turned on the magnetic field. Again: The *change* in magnetic dipole moment would still be directed opposite to the external field.
In conclusion: An external magnetic field influences the orbital motion in the atom in such a way that the *induced* magnetic dipole moment, i.e., the magnetic dipole moment associated with the change in the orbital motion, will be opposite to the external field. I.e., *diamagnetism*.