Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Spring 2005

Solution to øving 2

Guidance January 20. and 21.

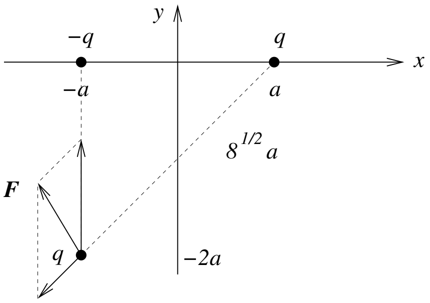
Exercise 1

a) **C**
Electrons have *negative* charge. An *excess* of N electrons thus implies a net *negative* charge:

$$Q = -Ne = -5 \cdot 10^{13} \cdot 1.6 \cdot 10^{-19}\ \text{C} = -8 \cdot 10^{-6}\ \text{C} = -8\ \mu\text{C}$$

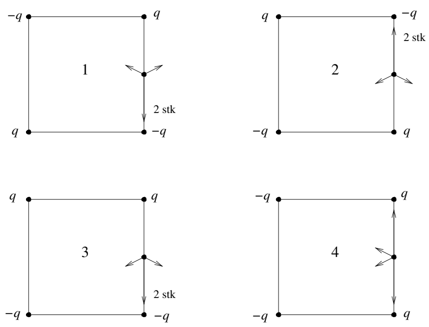
Here, μ denotes micro, i.e., $1\ \mu\text{C} = 10^{-6}$ C.

b) **A**
Here, it is sufficient to consider the direction of the partial forces that act on the third charge:



With Pytagoras, we have a distance of $\sqrt{8}a$ between the two positive charges. Since the Coulomb force is proportional to $1/r^2$, the force between the two positive charges becomes half the force between the negative and the positive charge. The vector sum becomes as indicated in the figure, i.e., a total force $\boldsymbol{F}$ with *negative* x component and *positive* y component.

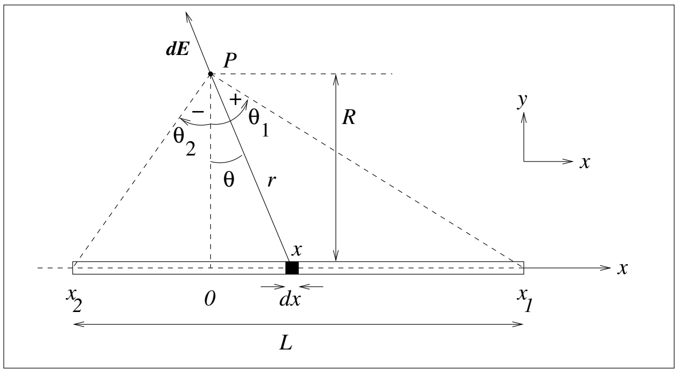
c) **C**
Total electric field in P is the vector sum of the contributions from the four point charges. The configuration in figure 3 yields the largest field strength. (No field contribution has a component upwards in this case.)



Exercise 2

a) With a line charge (i.e., charge pr unit length) λ, the charge dq and Q on a small length dx and on the whole rod becomes, respectively,

$$dq = \lambda\, dx \qquad Q = \lambda L$$



b) Electric field from element dx in position x:

$$d\boldsymbol{E} = \frac{\lambda\, dx}{4\pi\varepsilon_0 r^2}\hat{r} = A\frac{dx}{r^2}\hat{r}$$

where we have introduced $A = \lambda/4\pi\varepsilon_0$. From the figure we see that this vector has components

$$dE_x = -dE\sin\theta = -\frac{A\,dx}{r^2}\sin\theta \qquad dE_y = dE\cos\theta = \frac{A\,dx}{r^2}\cos\theta$$

Here, we have chosen $x = 0$ when $\theta = 0$, and the sign is in agreement with the given information, i.e., $\theta > 0$ when $x > 0$. We use the given hint and express dx and $1/r^2$ by the angle θ:

$$\begin{aligned} x &= R\tan\theta \;\Rightarrow\; dx = \frac{Rd\theta}{\cos^2\theta} \\ r &= \frac{R}{\cos\theta} \;\Rightarrow\; \frac{1}{r^2} = \frac{\cos^2\theta}{R^2} \\ \Rightarrow \frac{dx}{r^2} &= \frac{d\theta}{R} \end{aligned}$$

The components E_x and E_y of the field $\boldsymbol{E}$ in the point P from the whole rod is found by integrating dE_x and dE_y:

$$\begin{aligned} E_x &= \int dE_x = -\frac{A}{R}\int_{\theta_2}^{\theta_1}\sin\theta d\theta = \frac{A}{R}\Big|_{\theta_2}^{\theta_1}\cos\theta = \frac{\lambda}{4\pi\varepsilon_0 R}\left(\cos\theta_1 - \cos\theta_2\right) \\ E_y &= \int dE_y = \frac{A}{R}\int_{\theta_2}^{\theta_1}\cos\theta d\theta = \frac{A}{R}\Big|_{\theta_2}^{\theta_1}\sin\theta = \frac{\lambda}{4\pi\varepsilon_0 R}\left(\sin\theta_1 - \sin\theta_2\right) \end{aligned}$$

Comment: We might have been "unfortunate" and started with the relation $x = r\sin\theta$, which yields $dx = r\cos\theta\, d\theta + \sin\theta\, dr$, since both θ and r vary with x. But things work out nicely anyway: We have $\cos\theta = R/r$, i.e., $r = R/\cos\theta$, and hence

$$dr = -R\frac{1}{\cos^2\theta}(-\sin\theta)\,d\theta$$

so that

$$\begin{aligned}
\frac{dx}{r^2} &= \frac{r\cos\theta\,d\theta + \sin\theta\,dr}{r^2} \\
&= \frac{\cos\theta\,d\theta\cdot\cos\theta}{R} + \frac{\sin\theta\cdot R\sin\theta\,d\theta}{R^2} \\
&= \frac{d\theta}{R}\left(\cos^2\theta + \sin^2\theta\right) \\
&= \frac{d\theta}{R}
\end{aligned}$$

c) With P equally far from the two ends of the rod, we have $\theta_1 = -\theta_2$, and therefore $\cos\theta_1 - \cos\theta_2 = 0$ and $\sin\theta_1 - \sin\theta_2 = 2\sin\theta_1 = L/\sqrt{R^2 + L^2/4}$ Thus:

$$E_x = 0$$

and

$$E = E_y = \frac{\lambda L}{4\pi\varepsilon_0 R\sqrt{R^2 + L^2/4}}$$

Far away from the rod, i.e., $R \gg L$: We may then replace the square root by R because we may neglect $L^2/4$ in comparison with R^2. Then we obtain:

$$E \simeq \frac{\lambda L}{4\pi\varepsilon_0 R^2} = \frac{Q}{4\pi\varepsilon_0 R^2}$$

This is the same as the electric field from a point charge Q in a distance R. Not unexpected: If we are far away, the rod essentially looks like a point charge with total charge $Q = \lambda L$.

d) An infinitely long rod is achieved by letting $\theta_2 \to -\pi/2$ and $\theta_1 \to \pi/2$. Again we have $E_x = 0$, and hence

$$E = E_y = \frac{\lambda}{2\pi\varepsilon_0 R}$$

In other words: The field from an infinitely long line charge falls off inversely proportional with the distance R.

Exercise 3

a) The area of a thin ring with radius R and width dR is $dA = 2\pi R\,dR$, so the charge on such a ring is

$$dq = \sigma\,dA = 2\pi\sigma R\,dR$$

The area of the disk is $A = \pi R_0^2$, so the total charge on the disk is

$$Q = \sigma A = \pi\sigma R_0^2$$

If one does not remember the area of a circular disk, one may of course determine the total charge Q by integrating dq:

$$Q = \int dq = \int_0^{R_0} 2\pi\sigma R\,dR = 2\pi\sigma \left.\frac{R^2}{2}\right|_0^{R_0} = \pi\sigma R_0^2$$

And if one also does not remember what the circumference of a ring is, the charge on the thin ring may be found by starting with a small angle $d\phi$ and the area enclosed between R and $R + dR$. This area is $Rd\phi \cdot dR$, and if we integrate this expression over ϕ from 0 to 2π, we get $2\pi R\,dR$, which must then be the area of the thin ring with radius R and width dR.

b) We divide the disk into infinitesimally thin rings with width dR (see figure below). All points on the ring lie in the same distance r from the point on the z axis. Diagonally located points (or: areas dA) contribute to the field in such a way that the x and y components of the total field becomes zero (cf example in the lectures). The z component becomes:

$$dE_z = \frac{dQ}{4\pi\varepsilon_0 r^2}\cos\theta$$

Since r is constant around the whole ring, one may let dQ be the charge on the thin ring:

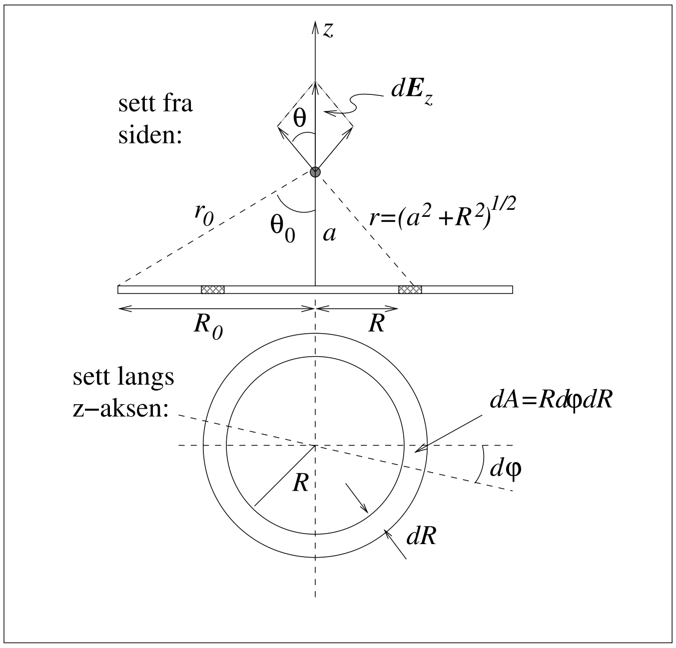
$$dQ = \sigma R\,dR \int_0^{2\pi} d\phi = 2\pi\sigma R\,dR$$

Hence the field from the disk becomes

$$E_z = \int dE_z = \frac{1}{4\pi\varepsilon_0}\int_0^{R_0} \frac{2\pi\sigma Ra\,dR}{(a^2+R^2)^{3/2}} = \frac{\sigma a}{2\varepsilon_0}\left.\frac{(-1)}{\sqrt{a^2+R^2}}\right|_0^{R_0} = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{a}{\sqrt{a^2+R_0^2}}\right)$$

Here, we have used

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2+R^2}}$$



An alternative would have been to use the angle θ as integration variable:

$$\begin{aligned}
\tan\theta &= \frac{R}{a} \;\Rightarrow\; d(\tan\theta) = \frac{d\theta}{\cos^2\theta} = \frac{dR}{a} \\
r &= \frac{a}{\cos\theta} \\
\int_0^{R_0} \frac{R\,dR}{r^2}\cos\theta &= \int_0^{\theta_0} \left(\frac{\cos\theta}{a}\right)^2 a\tan\theta \frac{a\,d\theta}{\cos^2\theta}\cos\theta = \int_0^{\theta_0} \sin\theta\,d\theta \\
&= 1-\cos\theta_0 = 1-\frac{a}{r_0} = 1-\frac{a}{\sqrt{a^2+R_0^2}}
\end{aligned}$$

where r_0 and θ_0 are defined in the figure above.

c) When $a \gg R_0$, one might at first (as in 2c above) consider replacing $\sqrt{a^2+R_0^2}$ with a. However, that only gives us the "trivial" solution $E_z = 0$, whereas we are interested in the dominating non-zero contribution to E_z. This means that we must expand $\sqrt{a^2+R_0^2}$ and include sufficiently many terms so that we end up with something which is no longer zero:

$$\begin{aligned}
E_z &= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{a}{a\sqrt{1+\frac{R_0^2}{a^2}}}\right) \\
&= \frac{\sigma}{2\varepsilon_0}\left(1-\left(1-\frac{R_0^2}{2a^2}+\cdots\right)\right) \\
&\simeq \frac{\sigma R_0^2}{4\varepsilon_0 a^2} \\
&= \frac{Q}{4\pi\varepsilon_0 a^2}
\end{aligned}$$

Here, we have used the approximation that was given in the text, $(1+\alpha)^{-1/2} \simeq 1-\alpha/2$, with $\alpha = R_0^2/a^2 \ll 1$.
This is the field in a distance a from a point charge $Q = \sigma A$, where $A = \pi R_0^2$ is the area of the disk. As expected: If we are sufficiently far away, we see no difference between a charged disk and a point charge.

In the opposite limit, $a \ll R_0$, we may neglect the term $a/\sqrt{a^2+R_0^2}$ in comparison with 1. This gives us

$$E_z = \frac{\sigma}{2\varepsilon_0}$$

In other words, a uniform electric field which depends neither on the distance a nor the extent R_0 of the disk. Hence, this is the field outside an *infinitely large* plane with surface charge density σ. It might not be obvious to you that the field then becomes independent of the distance to the plane, but that's the case! Of course, in practice we never have infinite planes of charge at our disposal, but this is nevertheless an important result: With a large charged plane, we generate an approximately uniform electric field near the plane, as long as we stay away from the edges of the plane. We will use this result many times later.